Math 128A, Spring 2007

Homework 11 Solution

1. Compute the local truncation error for the 3-Point Adams-Bashforth method.

   Solution:

   Recall that we obtained AB-3 by finding constants A, B, and C so that the approximation

   $$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} y'(t)dt \approx Ay'(t_n) + By'(t_{n-1}) + Cy'(t_{n-2})$$

   was exact for $y'(t)$ a polynomial of degree ≤ 2. We can use this to compute the local truncation error:

   If $p_2(t)$ is the interpolation of $y'(t)$ at t_n, t_{n-1}, and t_{n-2}, then we may write

   $$y'(t) = p_2(t) + e_2(t)$$

   and thus

   $$\int_{t_n}^{t_{n+1}} y'(t)dt = Ay'(t_n) + By'(t_{n-1}) + Cy'(t_{n-2}) + \int_{t_n}^{t_{n+1}} e_2(t)dt$$

   Using the Newton form for the error term, namely,

   $$e_2(t) = y'[t_{n-2}, t_{n-1}, t_n, t](t - t_{n-2})(t - t_{n-1})(t - t_n)$$

   We can apply the mean value theorem and integrate to get

   $$\int_{t_n}^{t_{n+1}} e_2(t)dt = y'[t_{n-2}, t_{n-1}, t_n, \bar{\xi}]\frac{h^4}{4}$$

   and then apply a theorem from class to get

   $$\int_{t_n}^{t_{n+1}} e_2(t)dt = \frac{y^5(\xi)}{4!}\frac{h^4}{4}$$

   putting this all together, we have

   $$y(t_{n+1}) - y(t_n) = Ay'(t_n) + By'(t_{n-1}) + Cy'(t_{n-2}) + \frac{y^5(\xi)}{4 \cdot 4!}h^4$$

   In particular, if $y(t_k) = y_k$ for $k = n, n-1, n-2$, then

   $$y(t_{n+1}) - y_{n+1} = \frac{y^5(\xi)}{4 \cdot 4!}h^4$$

   so we have found the local truncation error.

2. Derive the midpoint method and assess its stability.

   Solution:

   Applying the fundamental solution of calculus followed by the midpoint rule approximation,

   $$y(t_{n+1}) - y(t_{n-1}) = \int_{t_{n-1}}^{t_{n+1}} y'(t)dt \approx 2hy'(t_n)$$

   which leads to the multistep scheme

   $$y_{n+1} = y_{n-1} + 2hf(t_n, y_n)$$

   If we specialize this scheme to the ODE $y' = \lambda y$, it becomes

   $$y_{n+1} = y_{n-1} + 2h\lambda y_n$$

   This finite difference equation has characteristic polynomial

   $$r^2 - 2h\lambda r - 1$$

   with roots

   $$h\lambda \pm \sqrt{(h\lambda)^2 + 1}$$

   and hence general solution

   $$y_n = C_1(h\lambda + \sqrt{(h\lambda)^2 + 1})^n + C_2(h\lambda - \sqrt{(h\lambda)^2 + 1})^n$$

   Since some of the solutions diverge for any $h\lambda \neq 0$, this method is unstable.

3. Apply Backward Euler with step size $h = 0.5$ to approximate a solution to

   $$y' = -5y + 6\cos(x) + 4\sin(x) \qquad y(0) = 1$$

   on the interval $[0, 10]$.

   Solution:

   We need to solve the update rule

   $$y_{n+1} = y_n + (-5y_{n+1} + 6\cos(t_{n+1}) + 4\sin(t_{n+1}))$$

   for $y_n + 1$, i.e.:

   $$y_{n+1} = \frac{1}{4}[y_n + 6\cos(t_{n+1}) + 4\sin(t_{n+1})]$$

and then apply it 20 times so get

y_0	1.000000
y_1	2.045799
y_2	2.163374
y_3	1.644444
y_4	0.696188
y_5	−0.429196
y_6	−1.451168
y_7	−2.118260
y_8	−2.266833
y_9	−1.860432
y_{10}	−0.998539
y_{11}	0.107830
y_{12}	1.187797
y_{13}	1.976951
y_{14}	2.282078
y_{15}	2.028472
y_{16}	1.278226
y_{17}	0.215026
y_{18}	−0.900820
y_{19}	−1.796114
y_{20}	−2.251657

4. Use the finite difference method to get an approximate solution of $y" = ty(t) + t^2, y(0) = 1 = y(1)$, $h = 0.25$.

Solution:

We approximate $y(t)$ at 5 evenly spaced points in the interval $[0, 1]$ and approximate y''_n with $(y_{n+1} - 2y_n + y_{n-1})/h^2$. This leads to a system of equations

$$y_0 = 1$$
$$\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} = t_n y_n + t_n^2$$
$$y_5 = 1$$

with solution

y_0	1.0000000
y_1	0.3811011
y_2	0.0062476
y_3	0.1311988
y_4	1.0000000