Math 128A, Spring 2007

Homework 1 Solution

handout Evaluating the first 11 terms of the sequence given by

$$a_0 = 100\ln(101/100) \quad \text{and} \quad a_n = 100(1/n - a_{n-1})$$

yields

$$\begin{aligned}
\hat{a}_0 &= 0.995033\\
\hat{a}_1 &= 0.496691\\
\hat{a}_2 &= 0.330853\\
\hat{a}_3 &= 0.248017\\
\hat{a}_4 &= 0.198348\\
\hat{a}_5 &= 0.165241\\
\hat{a}_6 &= 0.142563\\
\hat{a}_7 &= 0.029448\\
\hat{a}_8 &= 9.555183\\
\hat{a}_9 &= -944.407156\\
\hat{a}_{10} &= 94450.715592
\end{aligned}$$

These diverge from $a_n = \int_0^1 100x^n/(100+x)dx$ because the computer is working with approximations of real numbers. When the computer approximates a_0, it produces a value $\hat{a}_0 = a_0 + \epsilon_0$, where ϵ_0 is some small error. If we feed this through the recurrence relation, we get

$$\hat{a}_n = a_n + (-100)^n \epsilon_0$$

so this small initial error is amplified by the recurrence relation.

3.1.10 Graphing $e^{-x} - \sin(x)$, it's easy to see that $[a,b] = [0, \pi/2]$ contains the smallest positive root. After n iterations, the error is bounded by

$$\frac{b-a}{2^{n+1}} = \frac{\pi}{2^{n+2}}$$

so it's sufficient to find n satisfying

$$\frac{\pi}{2^{n+2}} < 10^{-10}$$

I.e. $n > \log_2 \pi + 10\log_2 10 - 2 \approx 32.8708$.

3.2.2 Running the MATLAB code

```
x0 = 0; % initial guess
x1 = x0 - (x0^3-x0^2-x0-1) / (3*x0^2-2*x0-1);
while abs( x1 - x0 ) >= 10^-6
    x0 = x1;
    x1 = x0 - (x0^3-x0^2-x0-1) / (3*x0^2-2*x0-1)
end
x1
```

produces the output

```
x1 = 1.8393
```

3.2.3 (a) With $f(x) = x^2 - a$, the Newton update rule is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n}$$
$$= x_n - \frac{x_n^2}{2x_n} + \frac{a}{2x_n} = \frac{x_n}{2} + \frac{a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

(b) (3.19) says

$$\alpha - x_{n+1} = (\alpha - x_n)^2 \left[\frac{-f''(c_n)}{2f'(x_n)}\right]$$

For us, $\alpha = \sqrt{a}$, $f'(x) = 2x$, and $f''(x) = 2$. Hence

$$\sqrt{a} - x_{n+1} = (\sqrt{a} - x_n)^2 \left[\frac{-2}{4x_n}\right]$$

and rearranging a bit,

$$x_{n+1} - \sqrt{a} = \frac{1}{2x_n}(\sqrt{a} - x_n)^2$$

The relative error expression can be obtained by dividing by $\sqrt{a}$ and rearranging.

(c) Assuming $Rel(x_{n+1}) = (1/2)Rel(x_n)^2$,

$$\begin{aligned}
Rel(x_n) &= -(1/2)Rel(x_{n-1})^2 \\
&= -(1/2)(-(1/2)Rel(x_{n-2})^2)^2 \\
&= -(1/2)(-(1/2)(-(1/2)Rel(x_{n-3})^2)^2)^2 \\
&= -(1/2)^{1+2+4}Rel(x_{n-3})^8 \\
&= \cdots \\
&= -(1/2)^{1+2+4+\cdots+2^{n-1}}Rel(x_0)^{2^n} \\
&= -(1/2)^{2^n-1}Rel(x_0)^{2^n}
\end{aligned}$$

So if $Rel(x_0) = 0.1$ then $Rel(x_1)$, $Rel(x_2)$, $Rel(x_3)$, and $Rel(x_4)$ are -0.0050, -1.2500e-05, -7.8125e-11, and -3.0518e-21, respectively.