How to Gram-Schmidt Properly (i.e. without normalization)

Given vectors $u_0, u_1, u_2, u_3 \ldots$

define

$$v_0 = u_0$$

$$v_1 = u_1 - \frac{\langle u_1, v_0 \rangle}{\langle v_0, v_0 \rangle} v_0$$

$$v_2 = u_2 - \frac{\langle u_2, v_0 \rangle}{\langle v_0, v_0 \rangle} v_0 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$v_3 = u_3 - \frac{\langle u_3, v_0 \rangle}{\langle v_0, v_0 \rangle} v_0 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$$

$$\vdots$$

If you want Gaussian Quadrature
of degree of precision $2n+1$,
then what you do is
Gram Schmidt $1, x, x^2, x^3, \ldots, x^{n+1}$
with respect to $\langle \cdot , \cdot \rangle$ defined by
the weight function to get $p_0, p_1, p_2, \ldots, p_{n+1}$.

DO NOT normalize.

~~Take the~~ Let $c_0, c_1, c_2, \ldots, c_n$ be
the roots of p_{n+1}.

Then the quadrature formula is

$$\int_a^b f(x)\, w(x)\, dx \approx \sum_{k=0}^{n} A_k f(c_k)$$

where

$$A_k \stackrel{def}{=} \int_a^b L_k(x)\, w(x)\, dx$$

where L_k is the polynomial of degree $\le n$ s.t.
it is 1 on c_k and 0 on the other c's.

Context:

Suppose that $[a,b]$ is an interval.

Suppose that w is a nonnegative continuous function on $[a,b]$ which is positive for all but finitely many points.

Claim: For any fixed $n \in \mathbb{Z}_{\geq 0}$, we can find ~~[illegible]~~

$c_0, c_1, \ldots, c_n \in [a,b]$ and

$A_0, A_1, \ldots, A_n \in (0,\infty)$ such that

$\forall p \in P_{2n+1}$

$$\int_a^b p(x)\,w(x)\,dx = \sum_{k=0}^{n} A_k\, p(c_k).$$

Proof: We can define an inner product on the space of continuous functions $f : [a,b] \to \mathbb{R}$ as follows:

$$\langle f, g \rangle \stackrel{\text{def}}{=} \int_a^b f(x)\, g(x)\, w(x)\, dx .$$

This is bilinear and symmetric (why?).

Since w is nonnegative and positive for all but finitely many points and the functions are continuous, we see that

$$\langle f, f \rangle \stackrel{\text{def}}{=} \int_a^b [f(x)]^2 w(x)\, dx = 0$$

implies f is the constantly zero function, so $\langle \cdot, \cdot \rangle$ is positive-definite and thus is an inner product.

Now that we know we have an inner product, we can apply Gram–Schmidt to the functions

$x \mapsto 1,\ x \mapsto x,\ x \mapsto x^2, \dots, x \mapsto x^{n+1}$ on $[a,b]$

to get an orthogonal family of polynomial functions $p_0, p_1, p_2, \dots, p_{n+1}$ such that

$\forall k \in \{0,1,2,\dots,n+1\}$

$$\operatorname{span}\{p_0, p_1, \dots, p_k\} = P_k$$

and $\forall\, i, j \in \{0,1,2,\dots,n+1\}\quad i \neq j \Rightarrow$

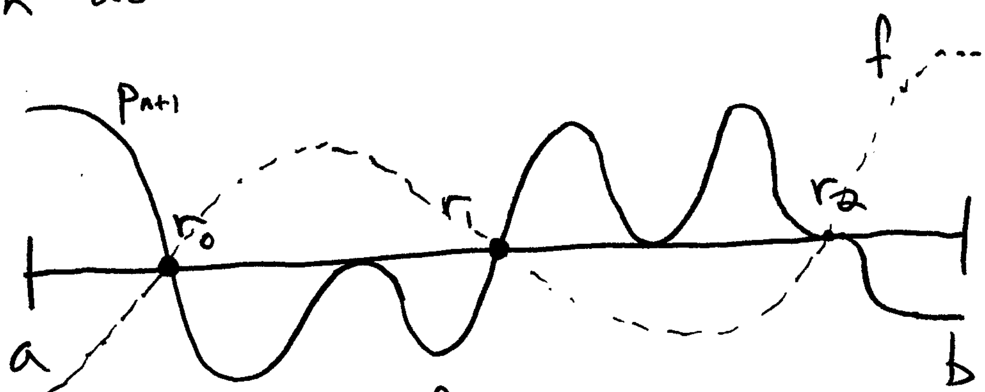
$$\langle p_i, p_j \rangle = 0.$$

It is not too hard to show (how?) that each p_j is a polynomial of degree j, so we might as well assume each p_j is monic.

Why do we care about monic? ↗

In particular, p_{n+1} has degree $n+1$, so it has $n+1$ complex roots counting multiplicities. But much more is true: in fact, p_{n+1} has $n+1$ distinct real roots, all which belong to (a,b). How can we prove this?

<u>Proof:</u> Let $r_0, r_1, \ldots, r_\ell$ be a list of those real roots of p_{n+1} that belong to (a,b) such that p_{n+1} changes sign across them.



Define the polynomial $f(x) \overset{\text{def}}{=} (x-r_0)(x-r_1)\cdots(x-r_\ell)$ so that $p_{n+1}\cdot f$ has constant sign on (a,b), or in the case that p_{n+1} has no such sign-changing roots in (a,b), define $f(x) \overset{\text{def}}{=} 1$ so that $p_{n+1}\cdot f$ has constant sign on (a,b).

Now assume, as ~~a~~ the beginning of a proof by contradiction, that $l < n$, i.e. that p_{n+1} does not have $n+1$ distinct sign-changing roots in (a,b).

You'll want this again later:

Then $\deg f \leq n$, so $f \in \text{span}\{p_0, p_1, \dots, p_n\}$ and thus $f = d_0 p_0 + d_1 p_1 + \dots + d_n p_n$ for some constants $d_0, d_1, \dots, d_n \in \mathbb{R}$, and thus

$p_{n+1} \perp \mathcal{P}_n$

$$\langle f, p_{n+1} \rangle = d_0 \langle p_0, p_{n+1} \rangle + d_1 \langle p_1, p_{n+1} \rangle + \dots + d_n \langle p_n, p_{n+1} \rangle = 0.$$

But this is a contradiction because $f\, p_{n+1}$ has constant sign (nonnegative, nonpositive) on (a,b) and is zero at finitely many points, so that by continuity

$$\langle f, p_{n+1} \rangle = \int_a^b \underbrace{f(x)\, p_{n+1}(x)\, w(x)}_{\text{constant sign}}\, dx \neq 0.$$

Thus p_{n+1} actually does have $n+1$ distinct sign-changing roots in (a,b).

Now that we know that p_{n+1} has $n+1$ distinct roots in (a,b), let us call them

$c_0, c_1, \ldots, c_n$.

We wish to find constants $A_0, \ldots, A_n$ such that for all $p \in P_{2n+1}$

$$\int_a^b p(x)w(x)\,dx = \sum_{k=0}^{n} A_k p(c_k).$$

Consider the Lagrange polynomials $L_0, L_1, \ldots, L_n$ which are the unique polynomials of degree $\leq n$ such that

$$L_j(c_k) = \delta_{jk} \text{ for all } j, k \in \{0,1,2,\ldots,n\}.$$

Certainly $\forall j \in \{0,1,\ldots,n\}$ $L_j \in P_{2n+1}$, and thus

$$\int_a^b L_j(x)w(x)\,dx = \sum_{k=0}^{n} A_k L_j(c_k) = \sum_{k=0}^{n} A_k \delta_{jk}$$

$$= A_j.$$ Thus we a forced to define

$$A_j \stackrel{\text{def}}{=} \int_a^b L_j(x)w(x)\,dx$$ if this method is to succeed.

Having defined $A_j = \int_a^b L_j(x)\, w(x)\, dx$,

we can prove that

$$\forall p \in P_n \quad \int_a^b p(x)\, w(x)\, dx = \sum_{k=0}^{n} A_k\, p(c_k).$$

This is not all that we want ($\forall p \in P_{2n+1}$), but it is going to get us further towards the goal.

<u>Proof</u>: $L_0, L_1, \dots, L_n$ are $n+1$ linearly independent polynomials in P_n, so they must form a basis for P_n since $\dim P_n = n+1$.

(why are they linearly independent?)

Suppose $p \in P_n$. Then, since $L_0, L_1, \dots, L_n$ span P_n, we can find constants $d_0, d_1, \dots, d_n \in \mathbb{R}$ s.t.

$$p = d_0 L_0 + d_1 L_1 + \dots + d_n L_n$$

(in fact, $d_k = p(c_k)$). Thus

$$\int_a^b p(x)\, w(x)\, dx = \sum_{k=0}^{n} \left(d_k \int_a^b L_k(x)\, w(x)\, dx \right) =$$

$$= \sum_{k=0}^{n} p(c_k)\, A_k = \sum_{k=0}^{n} A_k\, p(c_k). \quad \blacksquare$$

Now that we know that $\forall p \in P_n$

$$\int_a^b p(x)w(x)\,dx = \sum_{k=0}^{n} A_k\, p(c_k),$$ we would

like to prove the upgraded version:

$$\forall p \in P_{2n+1} \quad \int_a^b p(x)\,w(x)\,dx = \sum_{k=0}^{n} A_k\, p(c_k).$$

<u>Proof:</u> Let $p \in P_{2n+1}$ be arbitrary.

By the division algorithm, we can find

$q, r \in P_n$ such that

$p = p_{n+1} q + r$. Thus $\int_a^b p(x)w(x)\,dx =$

$$\int_a^b p_{n+1}(x)\,q(x)w(x)\,dx + \int_a^b r(x)w(x)\,dx =$$

$$= \underbrace{\langle p_{n+1}, q\rangle}_{=0} + \sum_{k=0}^{n} A_k\, r(c_k) = \sum_{k=0}^{n} A_k\, p(c_k)$$

since $q \in P_n$ (see earlier part labelled you'll want this later)

since $p(c_k) = \underbrace{p_{n+1}(c_k)}_{=0}\, q(c_k) + r(c_k) = r(c_k)$.

Now that we know that $\forall p \in P_{2n+1}$

$$\int_a^b p(x)\,w(x)\,dx = \sum_{k=0}^{n} A_k\, p(c_k),$$

we want to show that each $A_j > 0$.

<u>Proof</u>: L_j is continuous and $L_j(c_j) = 1$, and somewhere positive, so L_j^2 is nonnegative, continuous, and somewhere positive,

so $\int_a^b L_j^2(x)\,w(x)\,dx > 0$ since w is nonnegative and actually positive for all but finitely many points.

Thus

$$0 < \int_a^b L_j^2(x)\,w(x)\,dx = \sum_{k=0}^{n} A_k L_j^2(c_k)$$

$$= \sum_{k=0}^{n} A_k (\delta_{jk})^2 = A_j. \quad \blacksquare$$