Chaos: Finding a Horseshoe on the Beaches of Rio

Steve Smale [1]

1. Prologue

I would like to thank the National Research Council of Brasil (CNPq) for their support for me in the first half of 1960, and this article is a report to it on the work done with that support. This work was done while I was at Instituto da Matematica, Pura e Aplicada (IMPA) in Brasil.

At that time I also received support from the National Science Foundation (NSF) of the U.S.A. as a postdoctoral fellow, and this also serves as a report to that institution. The last is perhaps more important in view of questions raised about my having used U.S. taxpayer's money for this research done on the beaches of Rio. In fact none other than President Johnson's science adviser, Donald Hornig, wrote on this issue in 1968 in the widely circulated magazine "Science":

> "This blythe spirit leads mathematicians to seriously propose that the common man who pays the taxes ought to feel that mathematical creation should be supported with public funds on the beaches of Rio ..."

I am sorry that this report is late (36 years).

2. What is Chaos?

Chaos is a new science establishing unpredictability as fundamental on the level of common experience. A belief in Newtonian determinism, that the present state of the world determines the future precisely, dominated scientific thinking for two centuries. This credo was based on certain laws of physics, Newton's equations of motion, which have the property that the initial condition determines the solution for all time.

[1] written for a meeting in Rio de Janeiro, in March 1996, celebrating the 45 th anniversary of CNPq

At the beginning of this century the advent of quantum mechanics exposed the great delusion of determinism. At least on the level of electrons, protons, and atoms it was discovered that uncertainty prevailed.

However in spite of quantum mechanics, Newton's equations govern the behaviour of the solar system, the weather and many situations of everyday life. The quantum revolution left intact many deterministic dogmas. For example well after the second world war scientists held the belief that long range weather prediction would be successful when computer resources grew large enough.

As a consequence of the theory of chaos, the world now knows that the philosophy of determinism has become untenable on every level. The coin-flipping syndrome is pervasive.

Chaos contributes much more than extending the domain of indeterminancy, just as quantum mechanics did so more than half a century earlier. The deeper understanding of dynamics underlying the theory of chaos has shed light on every branch of science.

Chaos developed not with the discovery of new physical laws, but by a deeper analysis of the equations underlying Newtonian physics. Chaos is a scientific revolution based on mathematics. Deduction rather than induction is the methodology. Chaos takes the equations of Newton, analyses them with mathematics, and uses that analysis to establish the widespread unpredictability in the phenomena described by those equations. Via mathematics, one establishes the failure of Newtonian determinism by using Newton's own equations!

3. Flying Down to Rio

In 1956 I obtained a PhD in the new branch of mathematics called topology, sometimes called rubber sheet geometry, since a topologist is allowed to bend rigid objects. Taking up the position of instructor at the University of Chicago, I became good friends with Elon Lima, a student from Brazil who was working on a thesis, also in topology. My Brazilian connection was further established when Elon introduced me to a visiting professor from Rio de Janeiro, Mauricio Peixoto, whose ideas in dynamics were soon to affect my own work.

In the summer of 1958 we moved to the Institute for Advanced Study (IAS) in Princeton, New Jersey. This was the locale made famous as Einstein's workplace in America, with Robert Oppenheimer as its director. I was supposed to spend two years there with a NSF postdoctoral fellowship. However, Mauricio and Elon invited me to finish the second year in Rio de Janeiro. So Clara and I and our two children, Nat and Laura, left Princeton in December, 1959, to fly down to Rio.

The children were so young that most of our luggage consisted of diapers, but nevertheless we were able to realize an old ambition of seeing Latin America. After visiting the Panamian jungle, the four of us left Quito, Ecuador, Christmas of 1959,on the famous Andean railroad down into the port of Guayaquil. Soon we were flying into Rio de Janeiro, recovering from sicknesses we had acquired in Lima. I still remember vividly, arriving at night, going out several times trying to get milk for the crying children, and returning with a substitute as cream or yogurt. We later learned that, in Rio, milk was sold only in the morning, on the street. At that time Brasil was truly part of the "third world".

However our friends soon helped us settle down into Brasilian life. We arrived in Brasil just after a coup had been attempted by an air force colonel. He fled the country to take refuge in Argentina, and we were able to rent, from his wife, his luxurious 11-room apartment in the district of Rio called Leme. We also hired his two maids. The U.S. dollar went a long way in those days.

Sitting in our upper story garden veranda we could look across to the hill of the favela (called Babylonia) where Black Orpheus was filmed. In the hot humid evenings preceeding Carnaval, we would watch hundreds of the favela dwellers descend to dance the samba in the streets. Sometimes I would join their wild dancing which paraded for many miles.

In the front of our apartment, the opposite direction from the hill, lay the famous beach of Copacabana. I would spend my mornings on that wide, beautiful, sandy beach, swimming and body surfing. Also I took a pen and paper and would work on mathematics.

4. Mathematics on the Beach

Very quickly after our arrival in Rio, I found myself working on mathematical research. My host institution, Instituto da Matematica, Pura e Aplicada (IMPA), funded by the Brazilian government, provided a pleasant office and working environment. Just two years earlier IMPA had established itself in its own quarters, a small colonial building in the old section of Rio called Botafogo. There were no undergraduates and only a handful of graduate mathematics students. There were also a very few research mathematicians, notably Peixoto, Lima, and an analyst named Leopoldo Nachbin. Also there was a good math library. Thus I would typically spend my afternoons at IMPA, discussing topology with Elon, dynamics with Mauricio and browsing in the library. Mathematics research typically doesn't require much, the most important ingredients being a pad of paper and a ballpoint pen. In addition, some kind of library resources, and colleagues to query are helpful. I was satisfied.

Especially enjoyable were the times spent on the beach. My work was mostly scribbling down ideas and trying to see how arguments could be sequenced. Also I would sketch crude diagrams of geometric objects flowing through space, and try to link the pictures with formal deductions. Deeply involved in this kind of thinking and writing on a pad of paper, the distractions of the beach didn't bother me. Moreover, one could take time off from the research to swim.

The surf was an exiting challenge and even sometimes quite frightening. One time when Lima visited my "beach office", we entered the surf and were both caught in a current which took us out to sea. Other bathers shouted the good advice to swim parellel to the shore to a place where we could return. Elon subsequently told me that he had felt doomed. [It was 34 years later just before Carnaval, that once again those same beaches almost did me in. This time a special wave bounced me so hard on the sand it injured my wrist, tore my shoulder tendon, and then that same big wave carried me out to sea. I was lucky to get back using my good arm.]

5. Letter from America

At that time, as a topologist, I prided myself on a paper that I had just published in a different field called dynamics or ordinary differential equations. I was delighted with a conjecture in that paper which had as a consequence (in modern terminology) "chaos doesn't exist"!

This euphoria was soon shattered.

I received a letter from a M.I.T. mathematician named Norman Levinson, who had coauthored the main graduate text in ordinary differential equations. He was a scientist to be taken seriously.

Levinson wrote me of an earlier result of his which effectively contained a counterexample to my conjecture. His paper in turn was a clarification of extensive work of the pair of British mathematicians Mary Cartwright and J. L. Littlewood done during World War Two. Cartwright and Littlewood had been analysing some equations that arose in doing war-related studies involving radio waves. They had found unexpected and unusual behaviour of solutions of these equations. In fact Cartwright and Littlewood had proved mathematically that chaos could exist, even in equations that arose naturally in engineering. But the world wasn't ready to listen, and even today their important contributions to chaos theory are not well-known. I never met Littlewood, but in the mid-sixties, Dame Mary Cartwright invited me to high table at the women's college (Girton) at Cambridge where she presided.

I worked day and night to try to resolve the challenge to my beliefs that the letter posed. It was necessary to translate Levinson's analytic arguments into my own geometric way of thinking. At least in my own case, understanding mathematics doesn't come from reading or even listening. It comes from rethinking what I see or hear. I must redo the mathematics in the context of my particular background. And that background consists of many threads, some strong, some weak. My background is stronger in geometric analysis, but following a sequence of formulae gives me trouble. I tend to be slower than most mathematians to understand an argument. The mathematical literature is useful in that it provides clues, and one can often use these clues to put together a cogent picture. When I have reorganized the mathematics in my own terms, then I feel an understanding, not before.

In any case I eventually convinced myself that indeed Levinson was correct, and that my conjecture was wrong. Chaos was already implicit in the analyses of Cartwright and Littlewood! The paradox was resolved, I had guessed wrongly. But while learning that, I discovered the horseshoe!

6. The Horseshoe

The horseshoe is a natural consequence of a geometrical way of looking at the equations of Cartwright-Littlewood and Levinson. This geometrization gives a comprehension to the mechanism of chaos, and a causal link to the widespread unpredictability in dynamics.

Chaos is a phenomenon of dynamics, and dynamics is the time evolution of a set of states of nature. We will suppose that time is measured in discrete units as seconds or minutes. A state of nature will be idealized as a point in the two dimensional plane.

We will start by describing a non-chaotic linear example. The idea is to take a square and to study what happens to a point on this square in one unit of time, under a transformation to be described.

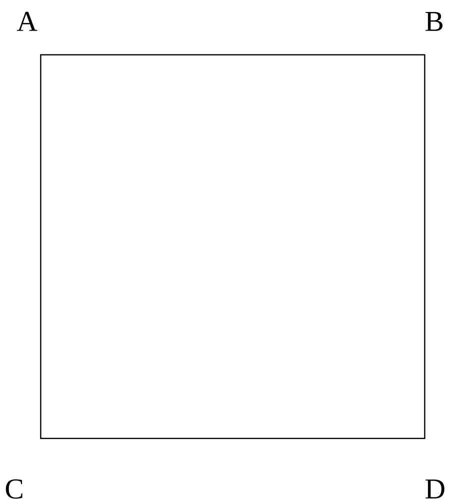

Figure 1:

The vertical dimension is now shrunk uniformly towards the center of the square and the horizontal is expanded uniformly at the same time. Using dots to outline the domain obtained by this process superimposed over the original square yields figure 2.

Here subscripts are used to denote the motion so that the corner A moves to A*. There is one point marked O which doesn't change. We have also shaded in the set of points which don't move out of the square in this process.

The second of our three stages in understanding is the perturbed linear example. Now the square is moved into a bent version of the elongated rectangle of figure 2. Thus figure 3 describes the motion of our square obtained by a small modification of figure 2.

The horseshoe is the fully non-linear version of what happens to points on the square, by an extension of the process expressed in figures 2 and 3. This is the situation when motion is expressed by a qualitative departure from the linear model. See Figure 4.

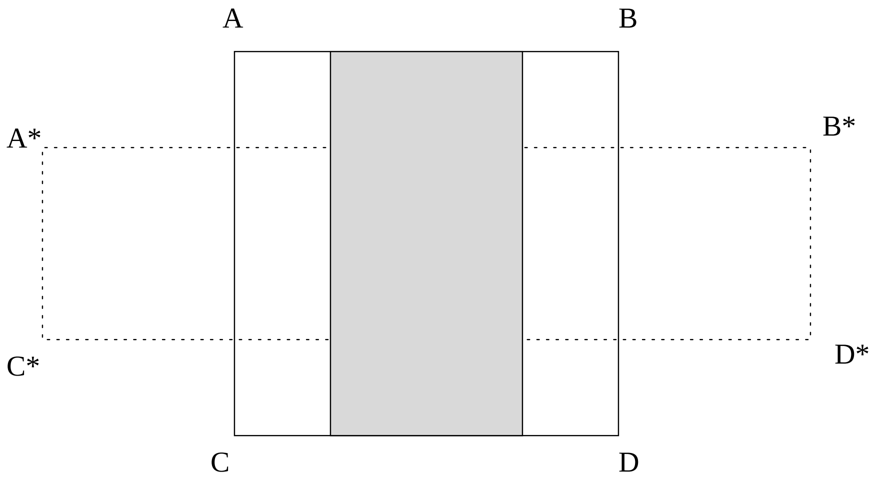

Figure 2:

The horseshoe is the domain surrounded by the dotted line.

Instead of a state of nature evolving according to a mathematical expression, the evolution is given geometrically. The full advantage of the geometrical point of view is beginning to appear. The more traditional way of dealing with dynamics was with the use of mathematical, e.g. algebraic, expressions. But a description given by formulae would be cumbersome. It would unlikely lead to insights or to a perceptive analysis, since that form of a description wouldn't communicate as efficiently the information in the figure. My background as a topologist, trained to bend objects as squares helped to make it possible to see the horseshoe.

The dynamics of the horseshoe is described by moving a point in the square to a point in the horseshoe according to Figure 4. Thus the corner marked A moves to the point marked A* in one unit of time.

The motion of a general point x in the square is a sequence of points x_0 x_1 x_2 ... Here $x_0 =$ x is the present state, x_1 is that state a unit of time later, x_2 that state two units of time later etc.

Now imagine our visual field to be just the square itself. When a point is moved out of the square we will discard that motion. The following Figure 5 shades in the points which don't leave the square in one unit of time.

We will call a motion which never leaves the square a visual motion. Our results in the next section concern visual motions.

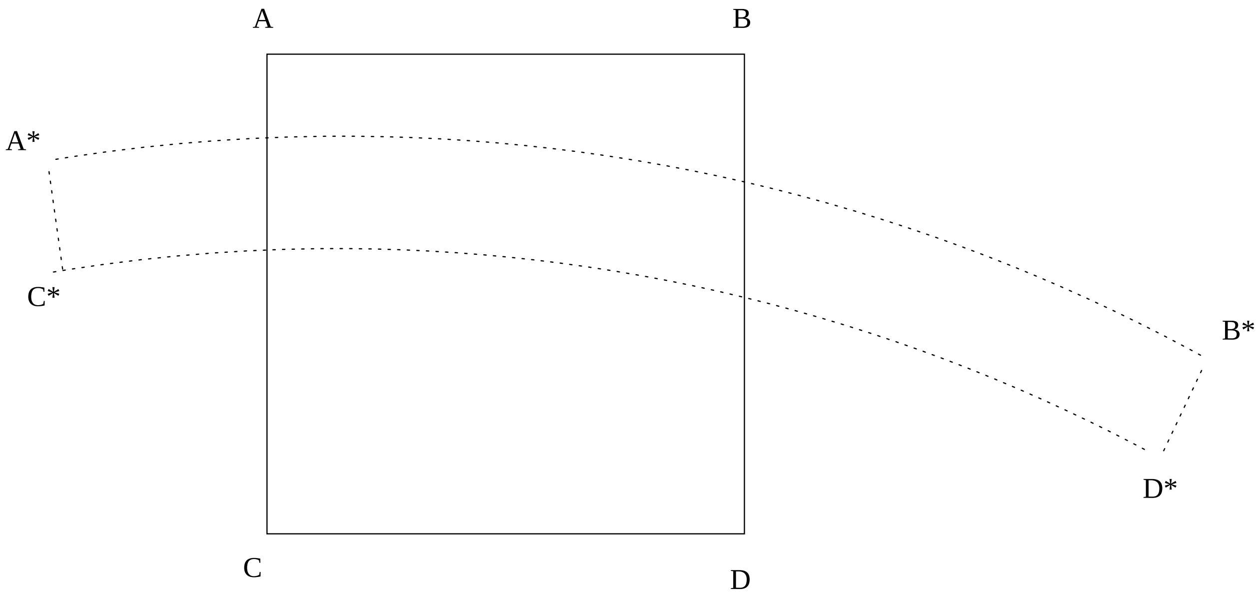

Figure 3:

In summary a fully non-linear motion finds its realization in the horseshoe. In the next section we will see the consequences to chaos that this picture carries.

7. The Horseshoe and Chaos: Coin Flipping

The laws of chance, with good reason, have traditionally been expressed in terms of flipping a coin. Whether heads or tails is the outcome of a coin toss is the paradigm of pure chance. On the other it is a deterministic process that governs the whole motion of the coin and hence the result, heads or tails, depends only on very subtle factors of the initiation of the toss.

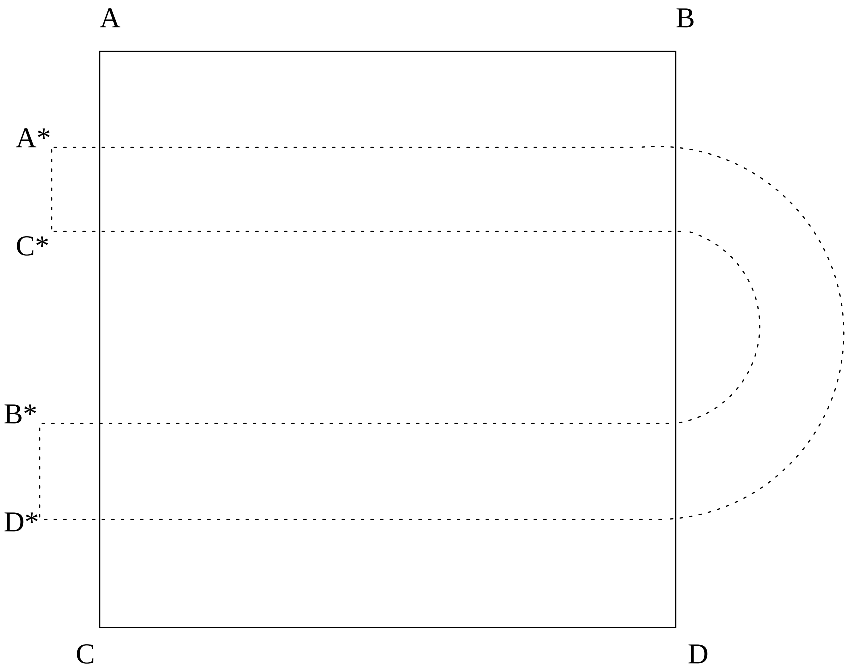

Figure 4:

A coin flipping experiment can be represented by a sequence, e.g. HTTHHTTTTH....., H for outcome heads, and T for outcome tails. [2] A general coin flipping experiment is thus a sequence $s_0\ s_1\ s_2$ ... where each s_i is either H or T.

Here is the result of the horseshoe analysis that I found on that Leme beach. Consider all the motions of the horseshoe construction which stay in the square, ie., don't drift out of our field of vision. These motions correspond precisely to the set of all coin flipping experiments! This discovery shows the occurence of unpredictability in fully non-linear motion and gives a mechanism of how determinism produces uncertainty.

The demonstration is based on the following construction. To each visual motion there is an associated coin flipping experiment. If $x_0\ x_1\ x_2$ ... is a visual motion of the horseshoe dynamics, at time i associate H or heads if x_i

[2]to give a complete picture in this section, one needs to reverse time and consider sequences of heads and tails which go back in time as well.

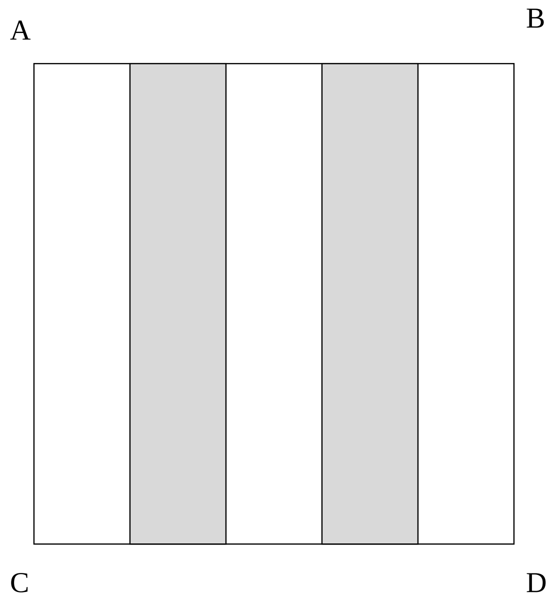

Figure 5:

lies in the top half of the square and T or tails if it lies in the bottom half.

Moreover, and this is the crux of the matter, every possible sequence of coin flips is represented by a horseshoe motion. Therefore the dynamics is as unpredictable as coin flipping. The following exhibits the correspondence in symbols.

$$x_0\ x_1\ x_2\ \ldots \text{ —— } s_0\ s_1\ s_2\ \ldots$$

Here $x_0\ x_1\ x_2\ \ldots$ is a motion lying in the square and $s_0\ s_1\ s_2\ \ldots$ is a sequence of H's and T's. On the left is a deterministically generated motion and on the right a coin flipping experiment.

The analysis shows that the above association is a complete natural identity between the set of motions lying in the square and the set of all coin flipping experiments. [3]

[3] mathematicians say that we have an isomorphism preserving the dynamics, or conjugacy, between the horseshoe motions lying in the square and the coin flipping experiments

The analysis is robust in that it doesn't change because of noise in the environment.

8. The Hidden Origins of Chaos

As chaos is a mathematically based revolution, it is not surprising to see that a mathematician first saw evidence of chaos in dynamics.

Henri Poincare was (with David Hilbert) one of the two foremost mathematicians in the world active at the end of the last century. Poincare was an originator of topology who had written an article claiming that a manifold with the same algebraic characteristics as the n-dimensional sphere was actually the n-dimensional sphere. When he found a mistake in his proof, restricting himself now to 3 dimensions, he formulated the assertion as a problem, now called Poincare's Conjecture. This problem is the biggest problem in topology and even one of the three or four great unsolved problems in mathematics today. What concerns us here however is this scientist's contribution to the theory of chaos.

Proving the solvability of the equations of celestial mechanics or the equations describing the motions of the planets was a celebrated problem at that time. Poincare in his extended studies in celestial mechanics was traumatized by a phenomena he discovered which showed the impossibility of ever solving such equations. He christened the concept "homoclinic point".

A homoclinic point is a motion tending to an equilibrium as time increases and also to that same equilibrium as time recedes into the past. The definition sounds harmless enough but carries amazing consequences. Poincare wrote concerning his discovery: [4]

> "One will be struck by the complexity of this figure which I won't even try to draw. Nothing can more clearly give an idea of the complexity of the three body problem and in general of all the problems of dynamics..."

In addition to showing the impossibility of solving the equations of planetary motion the homclinic point has turned out to be the trademark of chaos.

[4] my own translation from the French

G.D. Birkhoff, the most well known American mathematician before World War II, was heavily influenced by Poincare's work in dynamics. He developed these ideas and especially the properties of homoclinic points in his papers in the 20's and 30's.

I learned about homoclinic points and Poincare's work from browsing in Birkhoff's collected works which I found in IMPA's library.

Unfortunately, the scientific community had lost track of these important ideas surrounding the homoclinic points of Poincare. In the conferences in differential equations and dynamics that I attended at that time there was no awareness(if the conferees had known of this work of Poincare and Birkhoff, the conjectures in my rather publicized talks would have been answered earlier). Even Levinson never showed in his book, papers, or correspondence with me that he was aware of homoclinic points.

It is astounding how important scientific ideas can get lost, even when they are exposed by leading mathematicians of the preceeding decades.

It turned out, that soon after the discovery of the horseshoe, I found an important relation between horseshoes and homoclinic points. The existence of one of the two implies the existence of the other. Thus, in particular,in the presence of a homoclinic point, the trademark of chaos, one can produce a horseshoe. Thus the coin flipping syndrome underlies the homoclinic phenomenon, and helps to comprehend it.

9. The Third Force

I was lucky to find myself in Rio at the confluence of three different historical traditions in the subject of dynamics (called ordinary differential equations at that time). These three cultures while dealing with the same subject were isolated from each other, and this isolation inhhibited their development. We have already discussed two of these forces, Cartwright-Littlewood-Levinson and Poincare-Birkhoff.

The third had its roots in Russia with the school of differential equations of A. Andronov in Gorki in the 1930's. Andronov had died before the first time I went to the Soviet Union, but in Kiev, in 1961, I did meet his wife, Andronova-Leontovitch, who worked in differential equations as well.

Andronov had teamed up with the blind Soviet topologist Pontryagin to describe a geometric perspective of differential equations they called "rough", subsequently called structural stability. Chaos, in contrast to the two previously mentioned traditions, was absent in this development because of the restricted class of dynamics studied . Twenty years later the great American topologist Solomon Lefschetz became enthusiastic about Andronov-Pontryagin's work. It was through Lefschetz's influence, in particular, via an article of his student, De Baggis, that Mauricio Peixoto in Brazil, learned of structural stability.

Peixoto came to Princeton to work with Lefschetz in 1957 and this is the background of Mauricio's invitation to me.

I believe that it was the topologist's,Pontryagin and Lefschetz, hand in the subject that contributed to the fact that I was ready to listen to Mauricio.

10. Epilogue

Sometimes a horseshoe is considered an omen of good luck. The horseshoe I found on the beach of Rio certainly seemed to have such a property.

In that spring of 1960 I was primarily a topologist, mainly motivated by the problems of that subject, and most of all driven by the great unsolved problem posed by Poincare. Since I had started doing research in mathematics, I produced false proofs of the 3-dimensional Poincare Conjecture, returning again and again to that problem.

Now on those beaches, within two months of finding the horseshoe, I found to my amazement an idea which seemed to succeed provided I returned to Poincare's original assertion and then restricted the dimension to 5 or more. In fact the idea produced not only a solution of Poincare's Conjecture in dimensions greater than 4, but it gave rise to a large number of other nice results in topology. It was for this work that I received the Fields Medal in 1966.

There is more to finish the story. The Vietnam War was escalated by President Johnson in 1965, and I was moved to establish with Jerry Rubin a confrontational anti-war force. Our organization, the Vietnam Day Committee (VDC), with its teach-in, its troop train demonstrations and big marches, put me onto the front pages of the newspapers. These events led to a supoena by the House Unamerican Activities Committee (HUAC), which was issued while I was enroute to Moscow to receive the Fields Medal. The subsequent press conference I held in Moscow condemning the Vietnam War (as well as Russian intervention in Hungary) created a long lasting furore in Washington, DC. One outcome was Hornig's critique, quoted earlier, of my use of taxpayers funds to create mathematics on the beaches of Rio.

In this report I hope that I have communicated a bit of the mathematics that I created on those beaches at that time.

11. Reference

One has to be impressed by the omissions in this brief account. Some of the missing information may be found in:

From Topology to Computation: Proceedings of the Smalefest, editors, M. W. Hirsch, J. E. Marsden, M. Shub, Springer-Verlag, N. Y., 1993

City University of Hong Kong, April 1, 1996