RECURSIVE IN A GENERIC REAL

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Abstract. There is a comeager set $\mathcal{C}$ contained in the set of 1-generic reals and a first order structure $\mathfrak{M}$ such that for any real number X, there is an element of $\mathcal{C}$ which is recursive in X if and only if there is a presentation of $\mathfrak{M}$ which is recursive in X.

1. Introduction

The theory of a generic real is the theory of the almost everywhere behavior of all of the reals, and as such it can be well approximated. Consequently, constructions which can be implemented relative to any generic real can usually be simulated by approximation. For example, if a set of natural numbers X is recursive in every element of a co-meager set, then X is recursive. Similar statements for arithmetic in or constructible from are equally valid.

Counter to these observations, Slaman [1] produced a first order structure $\mathfrak{M}$ such that for all reals X, X is not recursive if and only if there is a presentation of $\mathfrak{M}$ which is recursive in X. X's computing a presentation of $\mathfrak{M}$ gives an existential criterion for determining whether X is not recursive. And so, there is something, the isomorphism type of $\mathfrak{M}$, which is common to all nonrecursive reals and which is not recursive. Wehner [2] independently produced an equivalent example formulated in terms of relatively recursive enumerations.

Now, we consider the question of whether there is an $\mathfrak{M}$ which is common exactly to generic reals. In sense of Theorem 1.1, the answer is yes.

Theorem 1.1. *For any uniformly Σ^0_2 family $\mathcal{D}$ of dense subsets of $2^{<\omega}$, there is a co-meager subset $\mathcal{C}$ of 2^ω and a countable model $\mathfrak{M}$ with the following properties.*

1. *If $G \in \mathcal{C}$, then G is $\mathcal{D}$-generic;*
2. *For all $X \subseteq \omega$, the following conditions are equivalent.*
   - (a) *There is a $G \in \mathcal{C}$ such that G is recursive in X.*
   - (b) *There is a presentation of $\mathfrak{M}$ which is recursive in X.*

For example, the family of dense sets that characterize 1-genericity is uniformly Σ^0_2, and so Theorem 1.1 applies to it.

While our proof of Theorem 1.1 is in the spirit of [1], it is quite different in detail. In the latter, one ensures that $\mathfrak{M}$ is not recursively presentable by ensuring that $\mathfrak{M}$ is not isomorphic to any recursive structure. There is a uniformly recursive approximation to the collection of recursively presented structures, so one has a countable diagonalization problem. Conversely, one constructs a functional Ψ so that if $\Psi(X)$ is not isomorphic to $\mathfrak{M}$ then the manner by which $\Psi(X)$ fails to duplicate $\mathfrak{M}$ provides the means to compute X.

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Here, we must ensure that for any real X, if $\mathfrak{M}$ is recursively presented relative to X, then there is a generic real which is recursive in X. So we have a coding problem: we must code the means to build a generic real within the isomorphism type of $\mathfrak{M}$; that is we must represent the means to meet dense sets within $\mathfrak{M}$. On the other hand, if $\mathfrak{M}$ contains nothing more than the means to meet dense sets, then generic reals should be able to represent $\mathfrak{M}$ since they meet dense sets themselves.

2. Trees associated with dense Σ^0_2 sets

2.1. **Dense functions.** For a nonempty set X and $n \in \omega$, X^n denotes the set of all sequences of elements from X of length n. And let

$$X^{<\omega} = \bigcup_{n\in\omega} X^n, \qquad X^{\leq n} = \bigcup_{i\leq n} X^i.$$

For each $u \in X^{<\omega}$, $|u|$ denotes the length of u, and for $i < |u|$, $u(i)$ denotes the i-th component of u. In this paper, $p, q, \ldots$ denote the elements of $2^{<\omega}$ and $\sigma, \tau, \ldots$ denote the elements of $\omega^{<\omega}$.

We say that a function $f : 2^{<\omega} \to 2^{<\omega}$ is *dense* if $p \subseteq f(p)$ for all p. For each Σ^0_2 dense subset D of $2^{<\omega}$, we can associate a Δ^0_2 dense function f_D with D so that $f_D(p) \in D$ for all $p \in 2^{<\omega}$. Furthermore, there is an effective procedure φ which produces a Δ^0_2 index of f_D from any Σ^0_2 index of D. Let $\mathcal{D} = \{D_n \mid n \in \omega\}$ be a given uniformly Σ^0_2 family of dense subsets of $2^{<\omega}$. Namely, each D_n is a dense subset of $2^{<\omega}$ and there is a recursive function π such that $\pi(n)$ gives a Σ^0_2 index of D_n. Then, the family $\mathcal{F} = \{f_{D_n} \mid n \in \omega\}$ is uniformly Δ^0_2 since $\varphi(\pi(n))$ gives a Δ^0_2 index of f_{D_n}. It is easy to see that

$$(\forall f \in \mathcal{F})(\exists p \in 2^{<\omega})[f(p) \subseteq G] \implies G \in 2^\omega \text{ is } \mathcal{D}\text{-generic.}$$

Hereafter, we will consider uniformly Δ^0_2 families of dense functions instead of the uniformly Σ^0_2 families of dense sets, For a uniformly Δ^0_2 family $\mathcal{F}$ of dense functions, we say that a real G is *$\mathcal{F}$-generic* if G satisfies $(\forall f \in \mathcal{F})(\exists p \in 2^{<\omega})[f(p) \subseteq G]$. Then, in order to prove Theorem 1.1, it is sufficient to show the following.

Theorem 2.1. *For any uniformly Δ^0_2 family $\mathcal{F}$ of dense functions, there is a comeager subset $\mathcal{C}$ of 2^ω and a countable model $\mathfrak{M}$ with the following properties.*

(1) *If $G \in \mathcal{C}$, then G is $\mathcal{F}$-generic;*
(2) *For all $X \subseteq \omega$, the following conditions are equivalent.*
   (a) *There is a $G \in \mathcal{C}$ such that G is recursive in X.*
   (b) *There is a presentation of $\mathfrak{M}$ which is recursive in X.*

2.2. **Recursive approximation.** In the following, we will be approximating a variety of sets and functions. We will use the suffix $[s]$ to indicate these quantities as they are approximated by stage s.

Let $f : 2^{<\omega} \to 2^{<\omega}$ be a Δ^0_2 dense function. By the limit lemma, there is a recursive approximation to f such that for all p

$$f(p) = \lim_{s\to\infty} f(p)[s].$$

We may assume that this approximation satisfies the following conditions.

(D1) if $s \geq n$ and $|p| = n$, then $f(p)[s] \in 2^{\leq s}$;
(D2) for all p and s, $p \subseteq f(p)[s]$.

Such an approximation is effectively obtained from f. That is, there is an effective procedure ψ which when given a Δ^0_2 index of f provides an index for the recursive approximation to f. We will fix a Δ^0_2 dense function f and its recursive approximation throughout this section.

2.3. **The tree $T(f,n)$.** Let σ be an element of $\omega^{<\omega}$. The *predecessor* of σ, $\mathrm{pd}(\sigma)$, is defined as $\mathrm{pd}(\sigma) = \sigma \upharpoonright (|\sigma| - 1)$ if $|\sigma| > 0$; and $\mathrm{pd}(\sigma) = \emptyset$ otherwise. Suppose $\sigma, \tau \in \omega^{<\omega}$. We say that σ is on the *left* of τ (or τ is on the *right* of σ) if there is an $i < \min(|\sigma|, |\tau|)$ such that $\sigma \upharpoonright i = \tau \upharpoonright i$ and $\sigma(i) < \tau(i)$.

Given a Δ^0_2 dense function f and an integer n, we will define a recursively enumerable tree $T(f,n)$ on ω by stages. Note that in the following, $T(f,n)[s]$ denotes the finite subtree of $T(f,n)$ which we have enumerated by stage s.

Stage 0. Enumerate $\emptyset$ into $T(f,n)$.

Stage s $(1 \leq s < n)$. Do nothing.

Stage n. Enumerate $\langle 2^n 3^n \rangle$ into $T(f,n)$.

Stage $s+1$ $(s \geq n)$. For $\sigma \in T(f,n)[s]$ with $\sigma = \langle x_1, x_2, \ldots, x_k \rangle$, we denote the first component of x_k by n_σ and the second component of x_k by s_σ. When $k = 0$ (i.e., $\sigma = \emptyset$), we let $n_\sigma = s_\sigma = 0$ for convenience. Suppose σ is a maximal element of $T(f,n)[s]$. We say that σ *fails to guess* f at stage $s+1$ if $s \geq s_\sigma$ and $(f \upharpoonright 2^{\leq n_\sigma})[s+1] \neq (f \upharpoonright 2^{\leq n_\sigma})[s_\sigma]$.

If there is no maximal element σ of $T(f,n)[s]$ which fails to guess f at stage $s+1$, then do nothing at this stage. Otherwise, take the rightmost element σ of $T(f,n)[s]$ which is maximal in $T(f,n)[s]$ and which fails to guess f at stage $s+1$, and do the following.

(1) For every maximal element τ of $T(f,n)[s]$ such that $n_\tau \geq n_\sigma$, enumerate $\tau^\frown\langle 2^{s+1} 3^{s+1} \rangle$ into $T(f,n)$. Thus, τ is not maximal in $T(f,n)[s+1]$.
(2) For every maximal element τ of $T(f,n)[s]$ such that $n_\tau = n_\sigma$, enumerate $\mathrm{pd}(\tau)^\frown\langle 2^{n_\sigma} 3^{s+1} \rangle$ into $T(f,n)$.

By construction, if $\sigma \in T(f,n)$, σ is enumerated at stage s_σ. Thus, the tree $T(f,n)$ is recursive. Further, if σ, τ are maximal in $T(f,n)[s]$ and $n_\sigma \leq n_\tau$, then we have $s_\sigma \leq s_\tau$.

Each maximal element of $T(f,n)[s]$ represents the stage s approximation of f. Suppose τ is maximal in $T(f,n)[s]$. We guess at stage s that $(f \upharpoonright 2^{\leq n_\tau})[s_\tau]$ is the correct value of $f \upharpoonright 2^{\leq n_\tau}$. If the guess fails at stage $s+1$, then we cancel it and start a new guess about f as follows. Take the rightmost σ which is maximal in $T(f,n)[s]$ and fails to guess f at stage $s+1$. If $n_\tau \geq n_\sigma$, then since τ fails to guess f at stage $s+1$, we start a new approximation of $f \upharpoonright 2^{\leq s+1}$ with $(f \upharpoonright 2^{\leq s+1})[s+1]$ by extending τ. At the same time, if $n_\tau = n_\sigma$, then we guess that the correct value of $f \upharpoonright 2^{\leq n_\tau}$ is $(f \upharpoonright 2^{\leq n_\tau})[s+1]$ by creating a new node on the immediate right of τ.

Figure 1 illustrates how the tree $T(f,n)$ grows when $n < s_1 < s_2 < s_3 < s_4$ and

$$(f \upharpoonright 2^{\leq n})[s_1] \neq (f \upharpoonright 2^{\leq n})[n], \tag{2.1}$$

$$(f \upharpoonright 2^{\leq s_1})[s_2] \neq (f \upharpoonright 2^{\leq s_1})[s_1], \tag{2.2}$$

$$(f \upharpoonright 2^{\leq s_2})[s_3] \neq (f \upharpoonright 2^{\leq s_2})[s_2], \tag{2.3}$$

$$(f \upharpoonright 2^{\leq n})[s_4] \neq (f \upharpoonright 2^{\leq n})[s_1]. \tag{2.4}$$

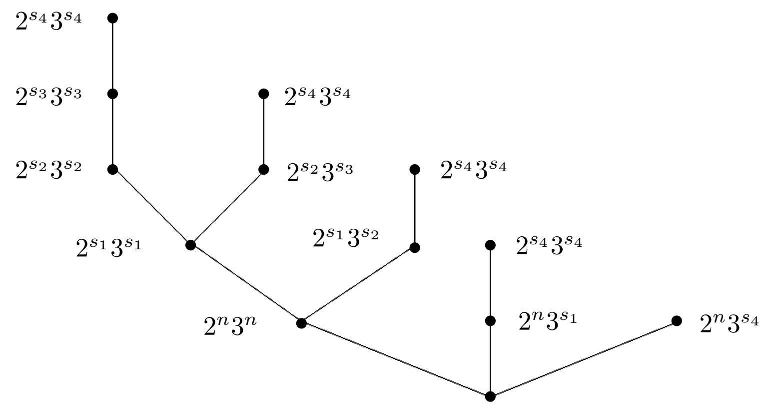

FIGURE 1

First, we guess that the correct value of $f \restriction 2^{\leq n}$ is $(f \restriction 2^{\leq n})[n]$. However, by (2.1), we see that this guess is incorrect at stage s_1. Thus, we change the guess about $f \restriction 2^{\leq n}$ to $(f \restriction 2^{\leq n})[s_1]$. And at the same time, based on this guess, we start the approximation of $f \restriction 2^{\leq s_1}$ with $(f \restriction 2^{\leq s_1})[s_1]$. But, then by (2.2), we see at stage s_2 that the approximation of $f \restriction 2^{\leq s_1}$ is not correct while the guess about $f \restriction 2^{\leq n}$ does not fail yet. Thus, we change the guess about $f \restriction 2^{\leq s_1}$ to $(f \restriction 2^{\leq s_1})[s_2]$ and start the approximation of $f \restriction 2^{\leq s_2}$ with $(f \restriction 2^{\leq s_2})[s_2]$ based on this guess. By (2.3), the approximation of $f \restriction 2^{\leq s_2}$ is seen to be incorrect at stage s_3. We thus change the guess about $f \restriction 2^{\leq s_2}$ to $(f \restriction 2^{\leq s_2})[s_3]$ and start the approximation of $f \restriction 2^{\leq s_3}$ with $(f \restriction 2^{\leq s_3})[s_3]$. Finally, by (2.4), we see at stage s_4 that the guess about $f \restriction 2^{\leq n}$ fails again. Therefore, we must cancel all of the approximations of f done so far, and start the approximation of $f \restriction 2^{\leq s_4}$ with $(f \restriction 2^{\leq s_4})[s_4]$ based on the guess that the correct value of $f \restriction 2^{\leq n}$ is $(f \restriction 2^{\leq n})[s_4]$.

2.4. **Terminal elements.** Given s and x, if there is a $t > s$ such that $(f \restriction 2^{\leq x})[t] \neq (f \restriction 2^{\leq x})[s]$, then we let $h^f(x, s)$ be the least such t. Otherwise, let $h^f(x, s) = s$.

Definition 2.2. (1) $d^f(x, 0) = x$, $\quad d^f(x, j+1) = h^f(x, d^f(x, j))$.
(2) $d^f(x) = \lim_{j\to\infty} d^f(x, j)$.
(3) $s_0^{\langle f,n\rangle} = n$, $\quad s_{i+1}^{\langle f,n\rangle} = d^f(s_i^{\langle f,n\rangle})$.
(4) For $i \geq 1$ and $j \geq 0$, $s_{i,j}^{\langle f,n\rangle} = d^f(s_{i-1}^{\langle f,n\rangle}, j)$.

In the following, we will drop the superscripts f and $\langle f, n\rangle$ for notational simplicity. By definition, for $i \geq 1$, we have

- $s_{i,0} = s_{i-1} \leq s_{i,1} \leq s_{i,2} \leq \ldots < \infty$,
- $s_i = \lim_{j\to\infty} s_{i,j}$,
- $f \restriction 2^{\leq s_{i-1}} = (f \restriction 2^{\leq s_{i-1}})[s_i]$.

We call an element of $T(f, n)$ *terminal* if it is maximal in $T(f, n)$. Since the approximation of $f \restriction 2^{\leq n}$ terminates at stage s_1, $\sigma_1 = \langle 2^n3^{s_1}\rangle$ is the rightmost terminal element of $T(f, n)$. If $s_1 = s_0 = n$, then σ_1 is never extended in $T(f, n)$ and σ_1 is the unique terminal element of $T(f, n)$. Otherwise, let j_1 be the least j such that $s_{1,j+1} = s_1$. Then, at stage s_1, $\langle 2^n3^{s_{1,j_1}}\rangle$ is extended by adding a new node with label $2^{s_1}3^{s_1}$ and the approximation of $f \restriction 2^{\leq s_1}$ starts with $f \restriction$

$2^{\leq s_1}[s_1]$. The approximation terminates at stage s_2 and $\sigma_2 = \langle 2^n 3^{s_{1,j_1}}, 2^{s_1}3^{s_2}\rangle$ is enumerated into $T(f,n)$ as a terminal element. σ_2 is the rightmost maximal element of $T(f,n)-\{\sigma_1\}$. In the case where $s_2 > s_1$, let j_2 be the least $j \geq s_1$ such that $s_{2,j+1} = s_2$. Then, $\langle 2^n 3^{s_{1,j_1}}, 2^{s_1}3^{s_{2,j_2}}\rangle$ is extended by adding a new node with label $2^{s_2}3^{s_2}$ and the approximation of $f \upharpoonright 2^{\leq s_2}$ starts with $(f \upharpoonright 2^{\leq s_2})[s_2]$. In the same manner, we see that $\sigma_3 = \langle 2^n 3^{s_{1,j_1}}, 2^{s_1}3^{s_{2,j_2}}, 2^{s_2}3^{s_3}\rangle$ is a terminal element of $T(f,n)$ and also the rightmost maximal element of $T(f,n) - \{\sigma_1, \sigma_2\}$. In general, we let σ_k be the rightmost maximal element of $T(f,n) - \{\sigma_1, \ldots, \sigma_{k-1}\}$. Then, σ_k is terminal in $T(f,n)$ and s_{σ_k} coincides with s_k. In view of this, we define the *approximation* of s_k at stage s, $s_k[s]$, as follows. First, take the list $\sigma_1, \ldots, \sigma_m$ of maximal elements of $T(f,n)[s]$ such that

- for each $k \geq 1$, σ_k is the rightmost element of $T(f,n)[s] - \{\sigma_1, \ldots, \sigma_{k-1}\}$;
- if $1 \leq k < m$, then $\mathrm{pd}(\sigma_k)$ has at least two extensions in $T(f,n)[s]$;
- σ_m is the unique extension of $\mathrm{pd}(\sigma_{m-1})$ in $T(f,n)[s] - \{\sigma_1, \ldots, \sigma_{m-1}\}$.

Then, for $1 \leq k \leq m$, we define $s_k[s] = s_{\sigma_k}$. For convenience, we set $s_0[s] = 0$. Note that $s_m[s] = s_{m-1}[s]$, and that if $s \geq s_k$ then $s_k[s] = s_k$.

2.5. **The relation R^G.** Let G be a given infinite subset of ω, and let P be a recursive infinite subset of ω. Let g_k denote the k-th element of G in increasing order.

$$G = \{g_0 < g_1 < \cdots < g_k < \cdots\}.$$

We define a G-recursively enumerable relation $R^G \subseteq P \times T(f,n)$ so that for each $p \in \mathrm{dom}(R^G)$, $\zeta^G(p) = \{\sigma \mid R^G(p,\sigma)\}$ is a maximal path in $T(f,n)$, and such that for each maximal finite path ζ in $T(f,n)$, there are infinitely many $p \in P$ such that $\zeta = \zeta^G(p)$. As before, $R^G[s]$ denotes the subset of R^G which we have enumerated by stage s.

Stage 0. Let p_0 be the least element of P and enumerate the pair $\langle p_0, \emptyset\rangle$ into R^G.

Stage $s+1$. For each maximal path ζ in $T(f,n)[s+1]$, pick a new element p from P and enumerate $\langle p, \tau\rangle$ into R^G for all $\tau \in \zeta$. This will ensure that infinitely many elements of P are associated with each maximal finite path in $T(f,n)$.

If $\zeta^G(p)[s] = \{\eta \in T(f,n)[s] \mid \langle p, \eta\rangle \in R^G[s]\}$ is not a maximal path in $T(f,n)[s+1]$, then let σ be its maximal element and find the smallest $k \geq 0$ such that $s_\sigma < s_k[s+1]$ and do the following.

Case 1. If $g_k < s_k[s+1]$, then pick the rightmost maximal path ζ in $T(f,n)[s+1]$ extending $\zeta^G(p)[s]$ and enumerate $\langle p, \tau\rangle$ into R^G for all $\tau \in \zeta$.

Case 2. If $g_k \geq s_k[s+1]$, then we wait until the construction of $T(f,n)$ reaches the stage g_k and associate p with the rightmost maximal path in $T(f,n)[g_k]$ which extends $\zeta^G(p)[s]$.

Lemma 2.3. *If $\{k \mid s_k \leq g_k\}$ is infinite, then $\zeta^G(p)$ is finite for every $p \in P$.*

Proof. Assume that $\{k \mid s_k \leq g_k\}$ and $\zeta^G(p)$ are both infinite. Then, the sequence $\{s_k\}_{k\in\omega}$ is strictly increasing. Take a sufficiently large $k \geq 2$ so that $s_k \leq g_k$ and p is enumerated into $\mathrm{dom}(R^G)$ at some stage $< s_{k-1}$. Let σ be a maximal element of $T(f,n)[s_{k-1}] \cap \zeta^G(p)$. Since σ is a maximal element of $T(f,n)[s_{k-1}]$ which is not terminal, it must be the case that $n_\sigma = s_\sigma = s_{k-1}$. Since $s_\sigma = s_{k-1} < s_k \leq g_k$, by the construction of R^G, $\zeta^G(p)$ takes the rightmost extension of σ in $T(f,n)[g_k]$, which is $\tau = \sigma^\frown\langle 2^{s_{k-1}}3^{s_k}\rangle$ and is terminal, a contradiction. Thus, we see that $\zeta^G(p)$ is finite. □

3. The model

Suppose $\widetilde{\mathfrak{F}}$ is a uniformly Δ^0_2 family of dense functions. We assume that a recursive enumeration of $\widetilde{\mathfrak{F}} \times \omega$ is fixed, and we denote the i-th element of $\widetilde{\mathfrak{F}} \times \omega$ by $\langle f^{(i)}, n^{(i)}\rangle$.

3.1. **The model $\mathfrak{M}(G)$.** Let $\mathfrak{L}$ consist of a constant symbol 0, unary function symbols s, t, and binary relation symbols R and $<_T$. Given an infinite subset $G = \{g_0 < g_1 < \cdots\}$ of ω, we define an $\mathfrak{L}$-structure $\mathfrak{M}(G)$ as follows.

- The constant 0 is interpreted by $\langle 0, \emptyset\rangle$.
- The function s is interpreted on the set $\{\langle i, \emptyset\rangle \mid i \in \omega\}$ by $\mathrm{s}^{\mathfrak{M}(G)}(\langle i, \emptyset\rangle) = \langle i+1, \emptyset\rangle$, and $\mathrm{s}^{\mathfrak{M}(G)}$ is identity on all points not in $\{\langle i, \emptyset\rangle \mid i \in \omega\}$. We use $\bar{i}$ to denote $s^i(0)$. Thus, $(\,\bar{i}\,)^{\mathfrak{M}(G)} = \langle i, \emptyset\rangle$.
- $<_T^{\mathfrak{M}(G)}$ is defined on $\bigcup_i(\{i\} \times T(f^{(i)}, n^{(i)}))$ as
$$\langle i, \sigma\rangle <_T^{\mathfrak{M}(G)} \langle j, \tau\rangle \iff i = j \ \&\ \sigma \subsetneq \tau.$$
- For $\sigma \in T(f^{(i)}, n^{(i)})$, we define $\mathrm{t}^{\mathfrak{M}(G)}(\langle i, \sigma\rangle) = (\,\overline{2^{n_\sigma}3^{s_\sigma}}\,)^{\mathfrak{M}(G)}$.
- Let $\{P_i\}_i$ be a recursive family of infinite disjoint subsets of ω, and $R_i^G \subseteq P_i \times T(D^{(i)}, n^{(i)})$ be the G-recursive relation defined as in the preceding section which picks out paths in $T(f^{(i)}, n^{(i)})$. Using $\{R_i^G\}_i$, we define $\mathrm{R}^{\mathfrak{M}(G)} \subseteq \bigcup_i P_i \times (\{i\} \times T(f^{(i)}, n^{(i)}))$ as follows.
$$\mathrm{R}^{\mathfrak{M}(G)}(p, \langle i, \sigma\rangle) \iff R_i^G(p, \sigma).$$

Finally, we let the universe of $\mathfrak{M}(G)$ be the set of all elements mentioned above. Since the above interpretations of $\mathfrak{L}$ is uniformly recursive in G, there is a recursive functional M such that $M(G)$ gives a presentation of $\mathfrak{M}(G)$.

3.2. **The model $\mathfrak{M}$.** From Lemma 2.3, we see that if G satisfies the condition

$$(*) \qquad \{k \mid s_k^{\langle f,n\rangle} \le g_k\} \text{ is infinite for all } \langle f, n\rangle \in \widetilde{\mathfrak{F}} \times \omega,$$

then for all $p \in \mathrm{dom}(R_i^G)$, $\zeta_i^G(p)$ is finite, where

$$\zeta_i^G(p) = \{\sigma \in T(f^{(i)}, n^{(i)}) \mid R_i^G(p, \sigma)\}.$$

Also, for every maximal finite path ζ in $T(f^{(i)}, n^{(i)})$, there are infinitely many elements p of P_i such that $\zeta = \zeta_i^G(p)$. From these facts, we obtain the following lemma.

Lemma 3.1. *If G_1 and G_2 satisfy the condition $(*)$, then $\mathfrak{M}(G_1)$ and $\mathfrak{M}(G_2)$ are isomorphic.*

Definition 3.2. Let $\mathfrak{M}$ be the isomorphism type of the models $\mathfrak{M}(G)$ which satisfy $(*)$.

4. Proof of Theorem 2.1

For a dense function f, let $\Gamma(f,n)$ denote the dense function associated with the dense set of all $p \in 2^{<\omega}$ such that there is a k for which the k-th element of $\{x \mid p(x)=1\}$ is defined and greater than or equal to $s_k^{f,n}$. Now let $\mathcal{F}$ be a uniformly Δ^0_2 family of dense functions on $2^{<\omega}$. Let $\widetilde{\mathcal{F}}$ be the closure of $\mathcal{F}$ under Γ. Namely, $\widetilde{\mathcal{F}}$ is the smallest family of dense functions such that

- $\mathcal{F} \subseteq \widetilde{\mathcal{F}}$;
- $(\forall f,n)[f \in \widetilde{\mathcal{F}} \implies \Gamma(f,n) \in \widetilde{\mathcal{F}}]$.

Then, it is easy to see that $\widetilde{\mathcal{F}}$ is also uniformly Δ^0_2.

We let $\mathcal{C}$ be the set of all $\widetilde{\mathcal{F}}$-generic reals, and define the models $\mathfrak{M}(G)$ and $\mathfrak{M}$ as in the preceding section. We will show that $\mathcal{C}$ and $\mathfrak{M}$ thus defined satisfy the conditions of Theorem 2.1.

The first condition (1) of Theorem 2.1 is trivial since each element of $\mathcal{C}$ is $\widetilde{\mathcal{F}}$-generic.

To prove (a) ⇒ (b) of (2), suppose G is recursive in X and $G \in \mathcal{C}$. Since G is $\widetilde{\mathcal{F}}$-generic and $\widetilde{\mathcal{F}}$ is closed under Γ, we see that $\mathfrak{M}(G) \simeq \mathfrak{M}$ by Lemma 2.3. Thus, $M(G)$ gives a presentation of $\mathfrak{M}$ which is recursive in G and hence recursive in X.

For the proof of (b) ⇒ (a), let Φ be a recursive functional and X be a real such that $\Phi(X) \simeq \mathfrak{M}$. We will construct a recursive functional Ψ so that $\Psi(X)$ is the characteristic function of a $\widetilde{\mathcal{F}}$-generic real.

For each $i \in \omega$, let $T_i^{\Phi(X)}$ denote the tree $\{x \mid \Phi(X) \models \bar{i} <_T x\}$. Since $T_i^{\Phi(X)} \simeq T(f^{(i)}, n^{(i)})$ via $\mathrm{t}^{\Phi(X)}$, where $\langle f^{(i)}, n^{(i)}\rangle$ is the i-th element of $\widetilde{\mathcal{F}} \times \omega$, we will not distinguish $T_i^{\Phi(X)}$ and $T(f^{(i)}, n^{(i)})$ hereafter. Also we denote by $R_i^{\Phi(X)}$ the relation $\{\langle p,x\rangle \mid x \in T_i^{\Phi(X)} \;\; \& \;\; \Phi(X) \models \mathrm{R}(p,x)\}$. Let f^m denote the m-th element of $\widetilde{\mathcal{F}}$. When $\langle f^m, n\rangle$ is equal to $\langle f^{(i)}, n^{(i)}\rangle$, we write $i(m,n)=i$. We may assume that the function $i(m,n)$ is recursive. We pick an element p_i recursively in X from $\mathrm{dom}(R_i^{\Phi(X)})$ for each i. The construction of $\Psi(X)$ proceeds by stages. At stage s, we will define an initial segment $\Psi(X)[s]$ of $\Psi(X)$ together with integers $u(s)$ and $n(m,s)$.

Stage 0. We set $\Psi(X)[0]=\emptyset$, $u(0)=0$ and $n(m,0)=0$ for all $m \in \omega$.

Stage $s+1$. Suppose that $\Psi(X)[s]$, $u(s)$ and $n(m,s)$ have been defined. Also suppose $\Psi(X)[s] \in 2^{\leq u(s)}$. If $u(s) \geq s+1$, then do nothing at this stage, namely, we just set $\Psi(X)[s+1]=\Psi(X)[s]$, $u(s+1)=u(s)$, and $n(m,s+1)=n(m,s)$ for all m.

Suppose $u(s) < s+1$. We say that f^m *requires attention* at stage $s+1$ if there is an element τ of $T(f^m,n)[s+1]$ which is a proper extension of σ where $n=n(m,s)$, $i=i(m,n)$ and σ is a maximal element of $T(f^m,n)[u(s)] \cap \zeta_i^{\Phi(X)}(p_i)$. In this case, we know that there is a proper extension of σ in $T(f^m,n)$ which is associated with p_i.

If there is no $m \leq s$ such that f^m requires attention at stage $s+1$, then do nothing at this stage. Otherwise, take the least m such that f^m requires attention at stage $s+1$. We let $u(s+1)$ be the first t such that $T(f^m, n(m,s))[t]$ contains a proper extension τ of σ such $p_{i(m,n(m,s))}$ is associated with τ by $R^{\Phi(X)}_{i(m,n(m,s))}$. We then set $\Psi(X)[s+1] = f^m[s_\tau](\Psi(X)[s])$. Note that $\Psi(X)[s+1] \in 2^{\leq u(s+1)}$ by

(D1). Finally, we define $n(j, s+1)$ as follows.

$$n(j, s+1) = \begin{cases} n(j, s), & \text{if } j \leq m; \\ s_\tau, & \text{if } j > m. \end{cases}$$

We say that f^m *acts* at this stage.

Verification. $\Psi(X)$ is constructed uniformly from $\Phi(X)$ and the elements of $\widetilde{\mathfrak{F}} \times \omega$, from which it follows that Ψ is a recursive functional. To see that $\Psi(X)$ is $\widetilde{\mathfrak{F}}$-generic, let f^m be the m-th element of $\widetilde{\mathfrak{F}}$. It is sufficient to show that there is an s such that $f^m(\Psi(X)[s]) = \Psi(X)[s+1]$. This is proved by finite injury argument.

First, we see, by induction on m, that every f^m requires attention only finitely often. Suppose that for every $k < m$, the number of stages where f^k requires attention is finite. Take a sufficiently large s_0 so that no f^k, $k < m$, requires attention at any stage after s_0. Then, $n(m, s)$ has constant value, say n, for every $s > s_0$. Let $i = i(m, n)$. In view of the construction, at each stage where f^m requires attention, we can find a new node of $T(f^m, n)$ with which p_i is associated. Since $\Phi(X) \simeq \mathfrak{M}$, the number of such nodes is finite. Thus, we see that f^m requires attention only finitely often.

Let $s+1$ be the last stage where f^m acts. If f^j, $j < m$, requires attention at some stage, then all actions so far for f^m are canceled at this stage and start to require attention. Thus, any of f^j, $j \leq m$, does not require attention at any stage after $s+1$. Let σ be the maximal element of $T(f^m, n)[u(s+1)] \cap \zeta_i^{\Phi(X)}(p_i)$. σ must be a terminal element of $T(f^m, n)$ since otherwise σ could be extended at later stage and f^m would require attention again, which is a contradiction. By the construction at stage $s+1$, we have

$$\Psi(X)[s+1] = f^m(\Psi(X)[s])[s_\sigma] = f^m(\Psi(X)[s]).$$

This complete the proof of Theorem 2.1.

5. Questions

5.1. Is there a countable model $\mathfrak{M}$ such that for all X, $\mathfrak{M}$ is recursively presented relative to X if and only some 1-generic real is recursive in X.

To remark on our first question, in our construction, we were given a family of dense functions $\mathfrak{F}$, we extended it to $\widetilde{\mathfrak{F}}$, and we produced a model $\mathfrak{M}$ such that for each real X, there is a presentation of $\mathfrak{M}$ which is recursive in X if and only if there is a $\widetilde{\mathfrak{F}}$ generic real G which is recursive in X. In brief, we used the fact that if G is $\widetilde{\mathfrak{F}}$ generic then G can compute a function which is not dominated by the functions which Skolemize the property that the functions in $\widetilde{\mathfrak{F}}$ are dense. This property may not hold for the original $\mathfrak{F}$.

5.2. Is there an analogous theorem with 'measure 1' in place of 'co-meager' in Theorem 1.1?

5.3. Can Δ^0_2 be improved in Theorem 2.1?

References

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