ON EXTENSIONS OF EMBEDDINGS INTO THE ENUMERATION DEGREES OF THE Σ^0_2-SETS

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Abstract. We give an algorithm for deciding whether an embedding of a finite partial order $\mathcal{P}$ into the enumeration degrees of the Σ^0_2-sets can always be extended to an embedding of a finite partial order $\mathcal{Q} \supset \mathcal{P}$.

1. The theorem. Reducibilities are relations on the power set of the natural numbers, conveying that a set $A \subseteq \omega$ can in some sense be "computed" or, in more generality, "defined", from another set $B \subseteq \omega$, usually denoted as $A \leq_r B$ (where r specifies what "computations" are allowed). Reducibilities are assumed to be reflexive and transitive but usually not antisymmetric, i. e., they give a partial preordering on $\mathcal{P}(\omega)$. They induce equivalence relations (defined by $A \equiv_r B$ iff $A \leq_r B$ and $B \leq_r A$), and the equivalence class of a set is called its *(r-)degree*. The degree of a set thus captures the computational complexity of a set of natural numbers while stripping away the information about the set irrelevant from a computational point of view (such as membership of a particular number, etc.).

The most important reducibility of classical computability theory is the *Turing reducibility*, denoting that a set A can be computed from a set B by means of an oracle Turing machine (i. e., by a hypothetical computer with unlimited resources and access to membership information about the "oracle" set B). In addition, there are various other reducibilities using models of computation which in various ways restrict the run time, memory space, or oracle access, or which allow infinite schemes of computation.

All the reducibilities mentioned in the above paragraph are based on the following model of computing a set A from a set B: A query about membership in A is reduced to an effectively generated sequence of queries about membership in B. *Enumeration reducibility* introduces a different concept: One produces an *enumeration* of a set A from an arbitrary *enumeration* of a set B, i. e., in the above Turing

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model, we use only *positive* information about membership in B and produce only *positive* membership information about A. More formally, we define the reducibility $A \leq_e B$ as follows:

Definition 1.1. Given sets $A, B \subseteq \omega$, we say A is *enumeration reducible* to B (denoted by $A \leq_e B$) if there is a computably enumerable set Φ of pairs (an *enumeration operator*) such that

$$A = \Phi(B) =_{\text{def}} \{x \mid \exists \langle x, F\rangle \in \Phi\,(F \text{ finite and } F \subseteq B)\}.$$

Note that this is closely modeled on Turing reducibility, since A is Turing reducible to B ($A \leq_T B$) if there is a computably enumerable set Φ of 4-tuples (the Turing functional) such that

$$A(x) = \Phi(B; x) = y \text{ iff } \exists \langle x, y, F, G\rangle \in \Phi\,(F,\, G \text{ finite and } F \subseteq B \text{ and } G \subseteq \overline{B}).$$

Enumeration reducibility is thus the natural analogue to Turing reducibility for "relative enumerability". It arises naturally in applications of computability theory to other areas of mathematics, such as for the analysis of types in effective model theory (see, e.g., Ash, Knight, Manasse, Slaman [AKMS89]) and in the study of existentially closed groups (see, e.g., Higman, Scott [HS88]).

The common analogue of both the computably enumerable Turing degrees and the Δ^0_2-Turing degrees in the enumeration degrees is the structure of the enumeration degrees of the Σ^0_2-sets, which coincide with the enumeration degrees of the sets enumeration reducible to $\mathbf{0}'_e$, the jump of the least enumeration degree, or, equivalently, the enumeration degree of $\overline{K}$. The Σ^0_2-enumeration degrees share some of the features of both the computably enumerable Turing degrees and of the Δ^0_2-Turing degrees: E.g., by Cooper [Co84], the Σ^0_2-enumeration degrees are dense (cf. the Sacks Density Theorem [Sa64] for the computably enumerable Turing degrees); but by a recent result of Lempp and Sorbi [LS02], the Σ^0_2-enumeration degrees admit the embedding of any finite lattice preserving least and greatest element (cf. Lerman, Shore [LS88] for the Δ^0_2-Turing degrees).

Slaman and Woodin [SW97] showed that the first-order theory of the partial order of the Σ^0_2-enumeration degrees is undecidable. An inspection of their proof shows that in fact undecidability appears after five alternations of quantifiers. However, the solution to the lattice embeddings problem by Lempp and Sorbi suggests that it may be easier to decide the $\forall\exists$-theory of the Σ^0_2-enumeration degrees than to decide the $\forall\exists$-theory of the computably enumerable Turing degrees.

Deciding the $\forall\exists$-theory amounts to deciding the following extension problem: Given finite partial orders $\mathcal{P}, \mathcal{Q}_0, \dots, \mathcal{Q}_n$ with $\mathcal{P} \subset \mathcal{Q}_i$ (for $i = 0, \dots, n$), decide whether any embedding of $\mathcal{P}$ into the Σ^0_2-enumeration degrees can be extended to an embedding of $\mathcal{Q}_i$ for some $i \leq n$ (where this i may depend on the particular embedding of $\mathcal{P}$). The classical extension of embeddings problem is a special case of this problem, namely, with $n = 0$.

In the present paper, we offer a solution to the extension of embeddings problem. Our analysis is closely modeled on the solution to the extension of embeddings problem for the computably enumerable Turing degrees by Slaman and Soare [SS01]. Before stating this result, we need some definitions:

Definition 1.2.

(1) A partial ordering $\mathcal{P}$ is *bounded* if it has a least element, 0, and a greatest element, 1.

(2) Given partial orderings $\mathcal{P} \subset \mathcal{Q}$ and a subset $S \subseteq \mathcal{Q}$, we set

$$\mathcal{A}(S) = \{a \in \mathcal{P} \mid \forall x \in S\,(a \geq x)\}, \text{ and}$$
$$\mathcal{B}(S) = \{a \in \mathcal{P} \mid \forall x \in S\,(a \leq x)\}.$$

We abbreviate $\mathcal{A}(x)$ and $\mathcal{B}(x)$ for $\mathcal{A}(\{x\})$ and $\mathcal{B}(\{x\})$, respectively. For simplicity, we often omit obvious parentheses thus writing for instance $\mathcal{BA}(S)$ instead of $\mathcal{B}(\mathcal{A}(S))$, etc.

(3) Given two subsets $S, T \subseteq \mathcal{Q}$, we also abbreviate by $S < T$ that

$$\forall x \in S\,\forall y \in T\,(x < y),$$

and analogously for $S \leq T$, $x < T$, etc.

In this notation, we can now state the result of Slaman and Soare:

Theorem 1.3 (Extension of Embeddings Theorem for the Computably Enumerable Turing Degrees) (Slaman, Soare [SS01])**.** *Given finite bounded partial orders $\mathcal{P} \subset \mathcal{Q}$, every embedding of $\mathcal{P}$ into the computably enumerable degrees extends to an embedding of $\mathcal{Q}$ if and only if neither of the following conditions holds:*

$$\exists x, y \in \mathcal{Q}\,(x \not\leq y \text{ and } \mathcal{BA}(x) \leq \mathcal{AB}(y)), \tag{1}$$
$$\exists y \in \mathcal{Q} - \mathcal{P}\,(\mathcal{Z}(y) \neq \emptyset \text{ and } \mathcal{BA}(\mathcal{Z}(y) \cup \mathcal{B}(y)) \not\subseteq \mathcal{B}(y)), \tag{2}$$

where

$$\mathcal{Z}(y) = \{z \in \mathcal{Q} - \mathcal{P} \mid z < y \text{ and } \mathcal{B}(y) \not\subseteq \mathcal{BA}(z)\}.$$

So, intuitively speaking, there are two types of obstacles to extensions of embeddings in the computably enumerable Turing degrees. The first is a lattice-theoretic one, since condition (1) above states that in some lattice extending $\mathcal{P}$, the infimum of the elements in $\mathcal{A}(x)$ is $\leq$ the supremum of the elements in $\mathcal{B}(y)$. The other obstacle is a saturation condition, stating that one cannot always extend an embedding to include a new point x above other new points z such that x bounds points in $\mathcal{P}$ which are "far" from the z's. (Later, in the extendibility construction, this situation would cause problems since the z's would be subject to a restraint coming from x that is too complicated for points in $\mathcal{P}$ bounding z to handle.) A typical instance of this latter obstacle is a partial ordering $\mathcal{P}$ consisting of 0, 1, and of two incomparable points a and b, extended to $\mathcal{Q}$ by inserting two points $z < x$ such that $z < a$ is incomparable with b, and $x > b$ is incomparable with a. Condition (2) above now applies since $\mathcal{Z}(x) = \{z\}$ and $\mathcal{BA}(\mathcal{Z}(x) \cup \mathcal{B}(x)) = \mathcal{P} \not\subseteq \{b, 0\} = \mathcal{B}(x)$. (We note here that the condition $\mathcal{Z}(x) \neq \emptyset$ in (2) above is necessary since $\mathcal{P} = \{0, c, 1\}$ and $\mathcal{Q} - \mathcal{P} = \{z, x\}$ with c incomparable to x, z and $z < x$ satisfy (2) without the condition $\mathcal{Z}(x) \neq \emptyset$, but clearly extendibility from $\mathcal{P}$ to $\mathcal{Q}$ holds.)

In the case of the Σ^0_2-enumeration degrees, we encounter an additional obstacle, which is already present in the case of one-point extensions. An instance of this obstacle was first found by Ahmad:

Theorem 1.4 (Ahmad (see Ahmad, Lachlan [AL98]))**.** *There are Σ^0_2-enumeration degrees $\mathbf{a}$ and $\mathbf{y}$ such that $\mathbf{a} \not\leq \mathbf{y}$ but any Σ^0_2-enumeration degree $\mathbf{x} < \mathbf{a}$ is $\leq \mathbf{y}$ (i. e., $\mathbf{a}$ itself is the only degree $\mathbf{x} \leq \mathbf{a}$ with $\mathbf{x} \not\leq \mathbf{y}$).*

The third obstacle (stated precisely in condition (3) in the theorem below) is a generalization of Ahmad's theorem, excluding the existence of extensions in the case where a new point $x \in \mathcal{Q} - \mathcal{P}$ is added incomparable to some $y \in \mathcal{Q}$ such that $\mathcal{B}(y) \supseteq \mathcal{B}(x)$ but $\mathcal{B}(y) \not\leq \mathcal{A}(x)$ where y may be in $\mathcal{P}$ or in $\mathcal{Q} - \mathcal{P}$.

The main theorem of this paper is thus the following

Theorem 1.5 (Extension of Embeddings Theorem for the Σ^0_2-Enumeration Degrees). *Given finite bounded partial orderings $\mathcal{P} \subseteq \mathcal{Q}$, every embedding of $\mathcal{P}$ into the Σ^0_2-enumeration degrees extends to an embedding of $\mathcal{Q}$ if and only if none of the following conditions holds:*

$$\exists x, y \in \mathcal{Q}\, (x \not\leq y \text{ and } \mathcal{BA}(x) \leq \mathcal{AB}(y)), \tag{1}$$

$$\exists y \in \mathcal{Q} - \mathcal{P}\, (\mathcal{Z}(y) \neq \emptyset \text{ and } \mathcal{BA}(\mathcal{Z}(y) \cup \mathcal{B}(y)) \not\subseteq \mathcal{B}(y)), \tag{2}$$

$$\exists x \in \mathcal{Q} - \mathcal{P}\, \exists y \in \mathcal{Q}\, (x \not\leq y \text{ and } \mathcal{B}(x) \subseteq \mathcal{B}(y) \text{ and } \mathcal{B}(y) \not\leq \mathcal{A}(x)), \tag{3}$$

where again

$$\mathcal{Z}(y) = \{z \in \mathcal{Q} - \mathcal{P} \mid z < y \text{ and } \mathcal{B}(y) \not\subseteq \mathcal{BA}(z)\}.$$

Note that Theorem 1.4 is a special case of condition (3) of Theorem 1.5, with $\mathcal{P} = \{0, a, y, 1\}$, $\mathcal{Q} - \mathcal{P} = \{x\}$, and $\mathcal{A}(x) = \{a, 1\}$.

Just as in Slaman and Soare [SS01], we immediately obtain the following

Corollary 1.6. *The extension of embeddings problem (in the language of bounded partial orderings) for the Σ^0_2-enumeration degrees is decidable.* □

We remind the reader that the extension of embeddings problem is only part of the $\forall\exists$-theory, for which decidability is still open:

Open Question 1.7. Is the $\forall\exists$-theory of the Σ^0_2-enumeration degrees in the language of partial ordering decidable?

We note that another fragment of the $\forall\exists$-theory of the Σ^0_2-enumeration degrees, the lattice embeddings problem, has recently also been shown to be decidable, in contrast to the situation for the computably enumerable Turing degrees, where it is still open:

Theorem 1.8 (Lattice Embeddings Theorem for the Σ^0_2-Enumeration Degrees) (Lempp and Sorbi [LS02])**.** *Every finite lattice can be embedded into the Σ^0_2-enumeration degrees, preserving least and greatest element.*

The rest of this paper is devoted to the proof of Theorem 1.5. We proceed in four parts, showing in sections 2, 3, and 4, respectively, that each of the conditions (1), (2), and (3) ensures non-extendibility; and in section 5 that extendibility holds otherwise.

Our notation is standard and generally follows Soare [So87], in particular its Chapter XIV.

2. The first non-extendibility theorem. In this section, we show that condition (1) of Theorem 1.5 ensures non-extendibility. We first state a lattice theoretic

Lemma 2.1. *Given a finite bounded partial ordering $\mathcal{P}$, there is a finite lattice $\mathcal{L}$ extending $\mathcal{P}$ as a partial ordering such that for any embedding e of $\mathcal{P}$ into a lattice $\mathcal{L}'$, e extends to a partial order embedding e' of $\mathcal{L}$ into $\mathcal{L}'$. Furthermore, for any subsets $A, B \subseteq \mathcal{P}$ such that A is upward closed and B is downward closed in $\mathcal{P}$ and such that $\mathcal{B}(A) \le \mathcal{A}(B)$, we have in $\mathcal{L}$ that*

$$\bigwedge A \le \bigvee B.$$

Proof. We first observe the following trivial facts about the operations $\mathcal{A}(X)$ and $\mathcal{B}(X)$ for any sets $X, Y \subseteq \mathcal{P}$:

$$X \subseteq Y \implies \mathcal{A}(X) \supseteq \mathcal{A}(Y) \text{ and } \mathcal{B}(X) \supseteq \mathcal{B}(Y), \tag{2.2}$$

$$X \subseteq \mathcal{AB}(X) \text{ and } X \subseteq \mathcal{BA}(X), \tag{2.3}$$

$$\mathcal{A}(X) = \mathcal{ABA}(X) \text{ and } \mathcal{B}(X) = \mathcal{BAB}(X). \tag{2.4}$$

We then define a closure operation on the power set of $\mathcal{P}$ by setting the *closure of* $X \subseteq \mathcal{P}$ to be $\mathcal{AB}(X)$. Then the lattice $\mathcal{L}$ is given by

$$\mathcal{L} = \{X \subseteq \mathcal{P} \mid X = \mathcal{AB}(X)\}, \tag{2.5}$$

with ordering and lattice operations on $\mathcal{L}$ defined by

$$X \preccurlyeq Y \Leftrightarrow X \supseteq Y, \tag{2.6}$$

$$X \vee Y = \mathcal{A}(\mathcal{B}(X) \cup \mathcal{B}(Y)), \tag{2.7}$$

$$X \wedge Y = \mathcal{A}(\mathcal{B}(X) \cap \mathcal{B}(Y)), \tag{2.8}$$

and the embedding of $\mathcal{P}$ into $\mathcal{L}$ given by

$$i(a) = \{b \in \mathcal{P} \mid b \ge a\}. \tag{2.9}$$

It is now easy to check, using (2.2)-(2.4), that $\mathcal{L}$ is a lattice and that i is a partial order embedding.

Now fix a partial order embedding e of $\mathcal{P}$ into any lattice $\mathcal{L}'$. We can then extend e to a partial order embedding e' of $\mathcal{L}$ into $\mathcal{L}'$ by setting

$$e'(X) = \bigwedge_{a \in X} e(a). \tag{2.10}$$

Clearly, e' preserves the ordering; it also preserves non-ordering (and is therefore injective). To see this, assume $X \not\preccurlyeq Y$. Suppose $\mathcal{B}(X) \subseteq \mathcal{B}(Y)$, then $\mathcal{AB}(X) \supseteq \mathcal{AB}(Y)$, which contradicts the fact that X and Y are closed. So fix some $x \in \mathcal{B}(X) - \mathcal{B}(Y)$ and some $y \in Y$ with $x \not\le y$. Now if $e'(X) \le e'(Y)$, then

$$e(x) \le \bigwedge_{a \in X} e(a) = e'(X) \le e'(Y) = \bigwedge_{b \in Y} e(b) \le e(y),$$

contradicting $x \not\le y$ and the fact that e preserves non-ordering.

Finally, fix subsets $A, B \subseteq \mathcal{P}$ such that A is upward closed and B is downward closed in $\mathcal{P}$ and such that $\mathcal{B}(A) \leq \mathcal{A}(B)$. Without loss of generality, we may assume that $A \in \mathcal{L}$ (otherwise replace A by $\mathcal{AB}(A)$ and note that $\mathcal{B}(A) = \mathcal{BAB}(A)$), and by (2.4), we have $\mathcal{A}(B) \in \mathcal{L}$. Now observe that $\mathcal{B}(A) \leq \mathcal{A}(B)$ and $A = \mathcal{AB}(A)$ imply $A \preccurlyeq \mathcal{A}(B)$ and that

$$A = \bigwedge_{a \in A} i(a) \text{ and } \mathcal{A}(B) = \bigvee_{b \in B} i(b).$$

This concludes the proof of Lemma 2.1. $\square$

It is now easy to see that condition (1) of Theorem 1.5 implies non-extendibility: Simply extend $\mathcal{P}$ to the finite lattice $\mathcal{L}$ given by Lemma 2.1, and fix an embedding of $\mathcal{L}$ into the Σ^0_2-enumeration degrees preserving least and greatest element given by Theorem 1.8. Then observe that Lemma 2.1 implies that all possible extensions meeting condition (1) of Theorem 1.5 are obstructed.

Remark 2.11. Note that the above proof actually shows a bit more than what Theorem 1.5 requires: It is possible to embed $\mathcal{P}$ into the Σ^0_2-enumeration degrees such that all one-point extensions of embeddings of $\mathcal{P}$ meeting condition (1) of Theorem 1.5 are obstructed *simultaneously*.

3. The second non-extendibility theorem. In this section, we show that condition (2) of Theorem 1.5 ensures non-extendibility.

We first fix some notation, roughly following Slaman and Soare [SS01, section 5]: Assume that we are given two finite posets $\mathcal{P} \subset \mathcal{Q}$ and an element $y \in Q - P$ with $\mathcal{Z}(y) \neq \emptyset$ and $\mathcal{BA}(\mathcal{Z}(y) \cup \mathcal{B}(y)) \not\subseteq \mathcal{B}(y)$. Let $\mathcal{Z}(y) = \{z_0, \ldots, z_k\}$, and for each $i \leq k$, let $\mathcal{A}(z_i) = \{a_{i,0}, \ldots, a_{i,k_i}\}$. Furthermore, let $\mathcal{B}(y) = \{b_0 = 0, b_1, \ldots, b_m\}$. Finally, fix $h \in \mathcal{BA}(\mathcal{Z}(y) \cup \mathcal{B}(y)) - \mathcal{B}(y)$; and for each $c \not\geq \mathcal{Z}(y)$, let i_c be the least $i \leq k$ such that $c \not\geq z_i$.

Our task is now to build an embedding of $\mathcal{P}$ into the Σ^0_2-enumeration degrees such that for any potential extension to $\mathcal{Q}$, one of the following holds:

(1) For some $i \leq k$ and some $j \leq k_i$, $Z_i \not\leq_e A_{i,j}$, contrary to $z_i \leq a_{i,j}$ in $\mathcal{Q}$;
(2) $H \leq_e \bigoplus_{l \leq m} B_l \oplus \bigoplus_{i \leq k} Z_i$; so $\bigoplus_{l \leq m} B_l \oplus \bigoplus_{i \leq k} Z_i \leq_e Y$ implies $H \leq_e Y$, contrary to $h \notin \mathcal{B}(y)$; or
(3) for some $c \notin \mathcal{A}(\mathcal{Z}(y))$, $Z_{i_c} \leq_e C$, contrary to the ordering in $\mathcal{Q}$.

We thus need to construct Σ^0_2-sets A (for each $a \in P$), satisfying the following *Requirements:* First of all, there are the "global" requirements:

$$\mathcal{C}^{d,c} : D \leq_e C$$

for each pair $d < c$ in $P - \{0, 1\}$;

$$\mathcal{C}^1 : C \equiv_e \overline{K}$$

for the element $c = 1$ in P; and

$$\mathcal{C}^0 : C \equiv_e \emptyset$$

for the element $c = 0$ in P.

We also need to satisfy the following "local" requirements. For all tuples of Σ^0_2-sets $(Z_i)_{i\le k}$ and all tuples of enumeration operators $(\Psi_{i,j})_{i\le k; j\le k_i}$, we satisfy

$$\mathcal{T}_{(Z_i)_{i\le k},(\Psi_{i,j})_{i\le k;j\le k_i}} : \forall i\le k\,\forall j\le k_i\,(Z_i=\Psi_{i,j}(A_{i,j})) \implies \exists\Gamma\,(H=\Gamma(\bigoplus_{l\le m} B_l\oplus\bigoplus_{i\le k} Z_i)) \text{ or } \exists\Delta\,\exists c\notin\mathcal{A}(\mathcal{Z}(y))(Z_{i_c}=\Delta(C)).$$

And finally, for all pairs $c,d\in P$ with $d\not\le c$ and all enumeration operators Θ, we satisfy the local requirements

$$\mathcal{I}^{d,c}_\Theta : D\ne\Theta(C).$$

Satisfying the global requirements $\mathcal{C}^{d,c}$ is simply achieved by coding D into C: Whenever a number x is chosen and targeted for D, it is chosen by stage x, and then $x\in D$ iff $x\in C$. Satisfying the global requirement $\mathcal{C}^0$ is achieved by setting $C=\emptyset$ for $c=0$. And the global requirement $\mathcal{C}^1$ is ensured by setting $C=\overline{K}\oplus\bigoplus_{d<c} D$ for $c=1$.

Satisfying one local requirement in isolation (in the presence of all the above global requirements) is simple: For a requirement $\mathcal{T}_{(Z_i)_{i\le k},(\Psi_{i,j})_{i\le k;j\le k_i}}$, we simply build the reduction Γ and keep it correct. For a requirement $\mathcal{I}^{d,c}_\Theta$, we appoint a witness n targeted for D and enumerate n into D. When (if ever) n enters $\Theta(C)$, we remove n from D and restrain C to keep $n\in\Theta(C)$. (Note here that $c\ne 1$ and $d\ne 0$, so this does not conflict with $\mathcal{C}^0$ or $\mathcal{C}^1$.)

Clearly, the strategy for an $\mathcal{I}$-requirement is finitary, so we only have to investigate the interaction between an $\mathcal{I}$-strategy and a finite number of higher-priority $\mathcal{T}$-strategies. We start by considering the interaction between one $\mathcal{I}$-requirement and one higher-priority $\mathcal{T}$-requirement. (We note one important difference here to the presentation in Slaman and Soare [SS01]: There, the "outcome of the $\mathcal{T}$-requirement along the true path" could "flip" twice, not just once as in the usual $\mathbf{0}'''$-constructions: First, it could switch from building Γ to building Δ, and then later from building Δ to discovering that one of the reductions $\Psi_{i,j}$ in the hypothesis of the $\mathcal{T}$-requirement is incorrect. This feature turns out to be unnecessary; a more careful analysis of the restraint reveals that in their, and in our, construction, one can simply restrain the sets involved so as to ensure that Δ-correction will not cause injury to any $\mathcal{I}$-strategies. We also correct a slight error in the description of the construction there, as noted in Slaman and Soare [SSta].) Our presentation here is somewhat reminiscent of the proof of the D.C.E. Nondensity Theorem [CHLLS91].

We first consider the case of an $\mathcal{I}^{d^0,c^0}_{\Theta^0}$-strategy (or $\mathcal{I}^0$-strategy, for short) below one $\mathcal{T}_{(Z_i)_{i\le k},(\Psi_{i,j})_{i\le k;j\le k_i}}$-strategy, building its functional $\Gamma(\bigoplus_{l\le m} B_l\oplus\bigoplus_{i\le k} Z_i)$, where we have to distinguish three cases:

Case A1: $c^0\not\ge\mathcal{B}(y)$: Then the $\mathcal{I}^0$-strategy needs to restrain only the sets B with $b\in\mathcal{B}(c)$, so the $\mathcal{T}$-strategy can correct its Γ via some set B with $b\in\mathcal{B}(y)-\mathcal{B}(c)$, and the $\mathcal{I}^0$-requirement can be satisfied finitarily.

Case A2: $c^0\ge\mathcal{B}(y)\cup\mathcal{Z}(y)$: Then $h\in\mathcal{BA}(\mathcal{Z}(y)\cup\mathcal{B}(y))$ implies $h\le c^0$ and thus $h\not\ge d^0$, so D^0-changes do not cause H-changes and so do not require Γ-correction via C^0; thus the $\mathcal{I}^0$-strategy can finitarily restrain C^0 without injury.

Case A3: $c^0\ge\mathcal{B}(y)$ but $c^0\not\ge\mathcal{Z}(y)$: In this case, possibly $h\ge d^0$, so a D^0-

for some $i \leq k$, or via B for some $b \in \mathcal{B}(y)$ with $b \leq c^0$. The latter may injure the Θ^0-computation. Thus, in the latter case, the $\mathcal{I}^0$-strategy will start destroying the reduction Γ and building a reduction $\Delta(C^0)$ computing $Z_{i_{c^0}}$, using larger and larger witnesses n^0. If this happens infinitely often, then the $\mathcal{I}^0$-strategy will make $\Gamma(\bigoplus_{l \leq m} B_l \oplus \bigoplus_{i \leq k} Z_i)$ finite and then has to ensure that $\Delta(C^0)$ indeed correctly computes $Z_{i_{c^0}}$ if the hypotheses of the $\mathcal{T}$-requirement hold, as explained in more detail below.

On the other hand, consider now an $\mathcal{I}^{d^1,c^1}_{\Theta^1}$-strategy (or $\mathcal{I}^1$-strategy, for short) below an $\mathcal{I}^0$-strategy, which destroys the functional $\Gamma(\bigoplus_{l \leq m} B_l \oplus \bigoplus_{i \leq k} Z_i)$ and builds a functional $\Delta(C^0)$ trying to compute $Z_{i_{c^0}}$. By the above analysis, $c^0 \geq \mathcal{B}(y)$ but $c^0 \not\geq \mathcal{Z}(y)$. We again distinguish cases and proceed as follows:

Case B1: $c^1 \not\geq c^0$: Then the $\mathcal{I}^1$-strategy will only restrain C^1 but not C^0. Thus $\Delta(C^0)$ can be corrected without injuring the C^1-restraint, and the $\mathcal{I}^1$-requirement can be satisfied finitarily.

Case B2: $c^1 \geq c^0$ and $c^1 \geq z_{i_{c^0}}$: Then $c^1 = a_{i_{c^0},j}$ for some j, so the $\mathcal{I}^1$-strategy can restrain $Z_{i_{c^0}}$ via C^1 and prevent Δ-correction by the $\mathcal{I}^0$-strategy. In order to enforce this restraint, we will ensure that the $\mathcal{T}$-strategy has already found the necessary computations from $A_{i_{c^0},j}$ so that the $\mathcal{I}^1$-strategy can prevent Δ-correction by C-restraint.

Case B3: $c^1 \geq c^0$ and $c^1 \not\geq z_{i_{c^0}}$: In this case, $Z_{i_{c^0}}$ may change even though C^0 and C^1 are restrained. However, note now that Δ-correction is never necessary, since the $\mathcal{I}^0$-strategy can satisfy its requirement finitarily without being injured by Γ-correction if $Z_{i_{c^0}}$ changes permanently and so the $\mathcal{I}^0$-strategy can restore a computation $\Theta^0(C^0; n^0)$, whereas, if $Z_{i_{c^0}}$ changes back, we may assume that the $\mathcal{T}$-strategy has already found computations of $Z_{i_{c^0}}$ from $A_{i_{c^0},j}$ for some $j \leq k_i$ with $a_{i_{c^0},j} \not\geq \mathcal{B}(y)$, and so the $\mathcal{I}^1$-strategy can permanently restrain $Z_{i_{c^0}}$ via $A_{i_{c^0},j}$ to prevent Δ-correction from ever injuring $\Theta^1(C^1; n^1)$.

An $\mathcal{I}$-strategy below several higher-priority $\mathcal{T}$-requirements acts in essentially the same way as above, but the nesting is a bit complicated, so we analyze some nontrivial cases for an $\mathcal{I}$-strategy below two higher-priority $\mathcal{T}$-requirements $\mathcal{T}^0$ and $\mathcal{T}^1$, say.

An $\mathcal{I}^{d^0,c^0}_{\Theta^0}$-strategy (or $\mathcal{I}^0$-strategy for short) below two $\mathcal{T}$-strategies, say, a $\mathcal{T}^0$-strategy building a functional Γ^0 and a $\mathcal{T}^1$-strategy building a functional Γ^1, must be finitary (as in Cases A1 and A2 above) unless $c^0 \geq \mathcal{B}(y)$ but $c^0 \not\geq \mathcal{Z}(y)$. In the latter case, the $\mathcal{I}^0$-strategy will act similarly to Case A3 above, destroying Γ^1 and building Δ^1 unless it eventually finds a computation $\Theta^0(C^0; n^0)$ cleared by both Γ^0 and Γ^1 (or no computation $\Theta^0(C^0; n^0)$ at all). So the $\mathcal{I}^0$-strategy has two possible outcomes: (1) destroying Γ^1 and building Δ^1 (the infinitary outcome), or (2) eventually finding a Γ^0- and Γ^1-cleared computation $\Theta^0(C^0; n^0)$ (or no computation $\Theta^0(C^0; n^0)$ at all), the finitary outcome. (Note that we destroy Γ^1 even if we find computations $\Theta^0(C^0; n^0)$ which are Γ^1- but not Γ^0-cleared.)

Next, we analyze the action of an $\mathcal{I}^{d^1,c^1}_{\Theta^1}$-strategy (or $\mathcal{I}^1$-strategy for short) below an $\mathcal{I}^0$-strategy building Δ^1. Again this is only interesting if $c^1 \geq \mathcal{B}(y)$ but $c^1 \not\geq \mathcal{Z}(y)$ (since otherwise the $\mathcal{I}^1$-strategy can deal with Γ^0 as in Case A1 or A2, essentially reducing the analysis to working below Δ^1 only). In addition, we may assume that $c^1 \geq c^0$ and $c^1 \not\geq z_{i_{c^0}}$ (since otherwise the $\mathcal{I}^1$-strategy can deal with Δ^1 as in Case B1 or B2, this time essentially reducing the analysis to working below Γ^0

only). On the other hand, assume that $c^1 \geq c^0$ and $c^1 \not\geq z_{i_{c^0}}$. Then a computation $\Theta^1(C^1; n^1)$ found by the $\mathcal{I}^1$-strategy is threatened by both Γ^0- and Δ^1-correction. Γ^0-correction is handled by destroying more of Γ^0 and Δ^1 and building Δ^0 as in Case A3 above. Against Δ^1-correction, we use the procedure of Case B3 above: The $\mathcal{I}^1$-strategy, upon finding a computation $\Theta^1(C^1; n^1)$, will first stop the $\mathcal{I}^0$-strategy and have it restore a computation $\Theta^0(C^0; n^0)$. If this computation $\Theta^0(C^0; n^0)$ is ever threatened by Γ^1-correction (and so Δ^1-correction is no longer necessary), then we may assume that the $\mathcal{T}^1$-strategy has already found enough computations of $Z^1_{i_{c^0}}$ from $A_{i_{c^0},j}$ (for some $j \leq k_{i_{c^0}}$ with $a_{i_{c^0},j} \not\geq \mathcal{B}(y)$) so as to make dangerous Δ^1-correction impossible from now on (while Γ^1 can still be corrected via some $b \not\leq a_{i_{c^0},j}$); now the $\mathcal{I}^1$-strategy can restrain $Z^1_{i_{c^0}}$ once and for all via $A_{i_{c^0},j}$, unless Γ^0-correction, and thus Γ^0-destruction, becomes necessary. (So again, there are two possible outcomes, an infinite one destroying Γ^0 and Δ^1 and building Δ^0, and a finite one, achieving $\Theta^1(C^1) \neq D^1$.)

On the other hand, below the infinite outcome of the $\mathcal{I}^1$-strategy, both Γ^1 and Δ^1 have been destroyed, so we first have to introduce a new $\mathcal{T}^1$-strategy building $\hat{\Gamma}^1$. Now we can analyze the action of an $\mathcal{I}^2$-strategy, say, dealing with an active Δ^0 and an active $\hat{\Gamma}^1$. Here again, for the interesting case, we will assume that $c^2 \geq \mathcal{B}(y)$ but $c^2 \not\geq \mathcal{Z}(y)$, as well as $c^2 \geq c^0$ and $c^2 \not\geq z_{i_{c^0}}$. Then the $\mathcal{I}^2$-strategy will look for a computation $\Theta^2(C^2; n^2)$, which is $\hat{\Gamma}^1$-cleared, or destroy more of $\hat{\Gamma}^1$ and build $\hat{\Delta}^1$ as in Case A3 above. If the $\mathcal{I}^2$-strategy does find a $\hat{\Gamma}^1$-cleared computation, then it will first stop the $\mathcal{I}^1$-strategy and have it restore a computation $\Theta^1(C^1; n^1)$ as in Case B3 above until, if ever, $Z^0_{i_{c^1}}$ changes back. If that happens, then we may assume that the $\mathcal{T}^0$-strategy has already found enough computations of $Z^0_{i_{c^1}}$ from $A_{i_{c^1},j}$ (for some $j \leq k_{i_{c^1}}$ with $a_{i_{c^1},j} \not\geq \mathcal{B}(y)$) so as to make dangerous Δ^0-correction impossible from now on (while Γ^0 can still be corrected via some $b \not\leq a_{i_{c^1},j}$); now the $\mathcal{I}^2$-strategy can restrain $Z^0_{i_{c^1}}$ once and for all via $A_{i_{c^1},j}$, the $\mathcal{I}^1$-strategy resumes its destruction of Γ^1 and Δ^1, and the $\mathcal{I}^2$-strategy will be satisfied finitarily.

Finally, we analyze the case of an $\mathcal{I}^3$-strategy, say, below an $\mathcal{I}^1$-strategy building Δ^0 and an $\mathcal{I}^2$-strategy building $\hat{\Delta}^1$. For the interesting case, assume that $c^2 \geq c^0, c^1$ and $c^2 \not\geq z_{i_{c^0}}, z_{i_{c^1}}$ (since otherwise the $\mathcal{I}^3$-strategy can deal with at least one of the Δ's as in Case B1 or B2 above, essentially reducing the analysis to working below at most one Δ). It is here that the minimal choice of the indices i_{c^0} and i_{c^1} turns out to be important: If $i_{c^0} \neq i_{c^1}$ then either $c^0 \geq z_{i_{c^1}}$ or $c^1 \geq z_{i_{c^0}}$, contradicting our assumptions from the previous sentence. Thus we conclude that $i_{c^0} = i_{c^1} = i^*$, say; and so the $\mathcal{I}^3$-strategy can choose the least $j^* \leq k_{i^*}$ with $a_{i^*,j^*} \not\geq \mathcal{B}(y)$, say, and $a_{i^*,j^*} \not\geq b^*$ for some $b^* \in \mathcal{B}(y)$. We now allow the $\mathcal{I}^3$-strategy to first stop the $\mathcal{I}^0$-strategy from correcting Δ^1 upon the $Z^1_{i^*}$-change caused by the $\mathcal{I}^3$-strategy's extracting n^3 from D^3, and let the $\mathcal{I}^0$-strategy restore its computation $\Theta^0(C^0; n^0)$ by restoring B for all $b \in \mathcal{B}(y)$. Then the $\mathcal{I}^0$-strategy will wait for Γ^0- or Γ^1-correction to destroy the computation $\Theta^0(C^0; n^0)$. (If this does not happen then the $\mathcal{I}^0$-strategy wins finitarily.) There are now two possibilities:

(1) Γ^1-correction first destroys $\Theta^0(C^0; n^0)$: Then $Z^1_{i^*}$ must have changed back, so we may assume that the $\mathcal{T}^1$-strategy has already found enough computations of $Z^1_{i^*}$ from A_{i^*,j^*} so that the $\mathcal{I}^1$-strategy can restrain $Z^1_{i^*}$ via A_{i^*,j^*} while allowing dangerous Γ^1-correction via B^*; and we let the $\mathcal{I}^0$-strategy resume its destruction of Γ^1 and its definition of Δ^1, stop the $\mathcal{T}^1$-strategy from destroying Γ^0 and correcting

Δ^0 upon the $Z^0_{i^*}$-change caused by the $\mathcal{I}^3$-strategy's extracting n^3 from D^3, and let the $\mathcal{I}^1$-strategy restore its computation $\Theta^1(C^1; n^1)$ by restoring B for all $b \in \mathcal{B}(y)$. If this computation is not later destroyed by Γ^0-correction then the $\mathcal{I}^1$-strategy wins finitarily; otherwise, $Z^0_{i^*}$ must have changed back also, so we may assume that the $\mathcal{T}^0$-strategy has already found enough computations of $Z^0_{i^*}$ from A_{i^*,j^*}; then the $\mathcal{I}^3$-strategy can restrain $Z^0_{i^*}$ via A_{i^*,j^*} while allowing dangerous Γ^0-correction via B^*; we let the $\mathcal{I}^1$- and $\mathcal{I}^2$-strategies also resume their destruction of Γ^0 and $\hat{\Gamma}^1$ and their definition of Δ^0 and $\hat{\Delta}^1$, and let the $\mathcal{I}^3$-strategy restore its computation $\Theta^3(C^3; n^3)$ by restoring B for all $b \in \mathcal{B}(y)$. This computation can now no longer be destroyed by any strategy above the $\mathcal{I}^3$-strategy.

(2) Γ^0-correction first destroys $\Theta^0(C^0; n^0)$: Then $Z^0_{i^*}$ must have changed back, so we may assume that the $\mathcal{T}^0$-strategy has already found enough computations of $Z^0_{i^*}$ from A_{i^*,j^*}; then the $\mathcal{I}^2$-strategy can restrain $Z^0_{i^*}$ via A_{i^*,j^*} while allowing dangerous Γ^0-correction via B^*; we let the $\mathcal{I}^0$- and $\mathcal{I}^1$-strategies resume their destruction of Γ^1 and Γ^0 and their definition of Δ^1 and Δ^0, stop the $\mathcal{I}^2$-strategy from destroying $\hat{\Gamma}^1$ and correcting $\hat{\Delta}^1$ upon the $Z^1_{i^*}$-change caused by the $\mathcal{I}^3$-strategy's extracting n^3 from D^3, and let the $\mathcal{I}^2$-strategy restore its computation $\Theta^2(C^2; n^2)$ by restoring B for all $b \in \mathcal{B}(y)$. If this computation is not later destroyed by $\hat{\Gamma}^1$-correction then the $\mathcal{I}^2$-strategy wins finitarily; otherwise, $Z^1_{i^*}$ must have changed back also, so we may assume that the $\mathcal{T}^1$-strategy has already found enough computations of $Z^1_{i^*}$ from A_{i^*,j^*}; then the $\mathcal{I}^3$-strategy can restrain $Z^1_{i^*}$ via A_{i^*,j^*} while allowing dangerous $\hat{\Gamma}^1$-correction via B^*; we let the $\mathcal{I}^2$-strategy also resume its destruction of $\hat{\Gamma}^1$ and its definition of $\hat{\Delta}^1$, and let the $\mathcal{I}^3$-strategy restore its computation $\Theta^3(C^3; n^3)$ by restoring B for all $b \in \mathcal{B}(y)$. This computation can now no longer be destroyed by any strategy above the $\mathcal{I}^3$-strategy.

Approximating the sets Z_i: We have so far glossed over one important point, namely, how we will approximate the sets Z_i for each $\mathcal{T}_{(Z_i)_{i \le k}, (\Psi_{i,j})_{i \le k; j \le k_i}}$-strategy. There are two potential problems which need to be addressed:

(1) We need to check that, for fixed $i \le k$, $\Psi_{i,j}(A_{i,j})$ is the same set for each $j \le k_i$. (If this fails, then we will ensure that the $\mathcal{T}_{(Z_i)_{i \le k}, (\Psi_{i,j})_{i \le k; j \le k_i}}$-strategy eventually has only the finite outcome and stops building its functional Γ.)

(2) Since each Z_i is part of the oracle for the corresponding functional Γ the correctness of which we need to ensure, we will also force our approximation to Z_i to be a Δ^0_2-approximation (and so each Z_i to be a Δ^0_2-set).

We achieve (1) and (2) by making all sets $A_{i,j}$ (for $a_{i,j} \not\geq \mathcal{B}(y)$) low, which will make $\Psi_{i,j}(A_{i,j})$ a Δ^0_2-set (and our approximation to it a Δ^0_2-approximation), and by slowing down the $\mathcal{T}$-strategy. For the lowness of each $A_{i,j}$ with $a_{i,j} \not\geq \mathcal{B}(y)$, we introduce extra

Lowness Requirements:

$$\mathcal{L}^a_{\Omega,x} : \exists^\infty s (x \in \Omega(A)[s]) \implies x \in \Omega(A)$$

for each $a \in P$ with $a \not\geq \mathcal{B}(y)$, each enumeration operator Ω, and each argument x. This is ensured as usual, by restraining a computation $x \in \Omega(A)$ found by an $\mathcal{L}^a_{\Omega,x}$-strategy ξ on the current approximation to the true path with the priority of the highest-priority $\mathcal{L}^a_{\Omega,x}$-strategy ξ' which has ever been eligible to act so far and then initializing any strategy $> \xi'$.

For each $i \le k$, there is now some (least) $j_i \le k_i$ such that $a_{i,j_i} \not\geq \mathcal{B}(y)$, and so $\Psi_{i,j_i}(A_{i,j_i})$ is a Δ^0_2-set with its Δ^0_2-approximation. We may now assume that

Z_i equals $\Psi_{i,j_i}(A_{i,j_i})$ (both in the limit and at every stage). We can then ensure that the sets $\Psi_{i,j}(A_{i,j})$ (for this i and all $j \leq k_i$) equal $\Psi_{i,j_i}(A_{i,j_i})$ by slowing down the $\mathcal{T}_{(Z_i)_{i\leq k},(\Psi_{i,j})_{i\leq k;j\leq k_i}}$-strategy. For this, we define *ξ-expansionary stages* by induction on stage s: Stage 0 is always ξ-expansionary. A stage $s > 0$ is ξ-expansionary if ξ is eligible to act at stage s (as defined below) and for all $x \leq$ the largest number mentioned by the greatest ξ-expansionary stage $s' < s$, we have

$$x \in Z_i[s] \implies \forall j \leq k_i\, (x \in \Psi_{i,j}(A_{i,j})[s]), \text{ and}$$
$$x \notin Z_i[s] \implies \forall j \leq k_i\, \exists s'' \in (s', s]\, (x \notin \Psi_{i,j}(A_{i,j})[s'']).$$

We then allow a $\mathcal{T}_{(Z_i)_{i\leq k},(\Psi_{i,j})_{i\leq k;j\leq k_i}}$-strategy ξ eligible to act at stage s to take the infinite outcome ∞ iff s is ξ-expansionary.

We are now ready to present the full construction, starting with the definition of the

Tree of strategies: Let $\Lambda = \{\infty <_\Lambda \mathit{fin}\}$ be the *set of outcomes*. (Intuitively, a $\mathcal{T}$-strategy will take outcome ∞ at expansionary, and outcome *fin* at non-expansionary stages. An $\mathcal{I}^{d,c}$-strategy can take the finitary outcome *fin*; in addition, if $c \geq \mathcal{B}(y)$ but $c \not\geq \mathcal{Z}(y)$ and there is some $\mathcal{T}$-requirement active along ξ, i.e., there is an active Γ threatening ξ, then ξ can also take outcome ∞. An $\mathcal{L}^a_{\Omega,x}$-strategy can only take outcome *fin*.)

We define the *tree of strategies* $T \subseteq \Lambda^{<\omega}$, and the assignment of requirements to nodes $\xi \in T$ (which we will call *strategies*), by induction on $|\xi|$.

First of all, fix an effective priority ordering of all $\mathcal{T}$-, $\mathcal{I}$-, and $\mathcal{L}$-requirements of order type ω. (So the global $\mathcal{C}$-requirements are not put on the tree T.)

Assign the highest-priority requirement from this list to the root $\emptyset \in T$, and call no requirement *active* or *satisfied along* the root $\emptyset$.

Given a node $\xi \in T$ such that a requirement has been assigned to ξ and for each $\mathcal{T}$-, $\mathcal{I}$-, and $\mathcal{L}$-requirement, we have determined whether it is active or satisfied along ξ, we now distinguish three cases depending on the type of requirement assigned to ξ:

Case 1: ξ has been assigned to $\mathcal{T}_{(Z_i)_{i\leq k},(\Psi_{i,j})_{i\leq k;j\leq k_i}}$: Then ξ has two immediate successors $\xi\hat{\ }\langle\infty\rangle$ and $\xi\hat{\ }\langle \mathit{fin}\rangle$ on T. We call $\mathcal{T}_{(Z_i)_{i\leq k},(\Psi_{i,j})_{i\leq k;j\leq k_i}}$ *active along* $\xi\hat{\ }\langle\infty\rangle$. We call $\mathcal{T}_{(Z_i)_{i\leq k},(\Psi_{i,j})_{i\leq k;j\leq k_i}}$ *satisfied along* $\xi\hat{\ }\langle \mathit{fin}\rangle$. We call all other requirements *active* or *satisfied along* $\xi\hat{\ }\langle o\rangle$ iff they are so along ξ (for $o \in \{\infty, \mathit{fin}\}$).

Case 2: ξ has been assigned to $\mathcal{I}^{d,c}_\Theta$: We distinguish two subcases:

Subcase 2.1: $c \not\geq \mathcal{B}(y)$, or $c \geq \mathcal{B}(y) \cup \mathcal{Z}(y)$, or there is no $\mathcal{T}$-requirement active along ξ: Then ξ has an immediate successor $\xi\hat{\ }\langle \mathit{fin}\rangle$ on T. We call $\mathcal{I}^{d,c}_\Theta$ *satisfied along* $\xi\hat{\ }\langle \mathit{fin}\rangle$, and call all other requirements *active* or *satisfied along* $\xi\hat{\ }\langle \mathit{fin}\rangle$ iff they are so along ξ.

Subcase 2.2: Otherwise: Let $\mathcal{T}'$ be the lowest-priority $\mathcal{T}$-requirement active along ξ. Then, in addition to the successor from Subcase 2.1, ξ also has an immediate successor $\xi\hat{\ }\langle\infty\rangle$. We call $\mathcal{T}'$ *satisfied along* $\xi\hat{\ }\langle\infty\rangle$, call all requirements of higher priority than $\mathcal{T}'$ *active* or *satisfied along* $\xi\hat{\ }\langle\infty\rangle$ iff they are so along ξ, and call no requirement of lower priority than $\mathcal{T}'$ *active* or *satisfied along* $\xi\hat{\ }\langle\infty\rangle$.

Case 3: ξ has been assigned to $\mathcal{L}^a_{\Omega,x}$: Then ξ has only one immediate successor $\xi\hat{\ }\langle \mathit{fin}\rangle$ on T. We call $\mathcal{L}^a_{\Omega,x}$ *satisfied along* $\xi\hat{\ }\langle \mathit{fin}\rangle$, and call all other requirements *active* or *satisfied along* $\xi\hat{\ }\langle \mathit{fin}\rangle$ iff they are so along ξ.

In all the above cases, we now *assign* to any immediate successor $\xi\hat{\ }\langle o\rangle \in T$ of ξ the highest-priority requirement not active or satisfied along $\xi\hat{\ }\langle o\rangle$.

(Again, the intuition here is that in Subcase 2.1, the $\mathcal{I}$-strategy always has finite outcome, whereas in Subcase 2.2, the $\mathcal{I}$-strategy may have finite outcome, or destroy the active functional Γ of a $\mathcal{T}$-strategy above and build a functional Δ instead. If the $\mathcal{I}$-strategy "flips" the outcome of a $\mathcal{T}$-requirement from "active" to "satisfied", then we have to start all over on all requirements of lower priority than that $\mathcal{T}$-requirement.)

We note an easy

Lemma 3.1. *For each $\mathcal{T}$-, $\mathcal{I}$-, or $\mathcal{L}$-requirement and each infinite path $p \in [T]$, there is a strategy $\xi \subset p$ such that the requirement is active (via ξ) along all ξ' with $\xi \subset \xi' \subset p$, or satisfied (via ξ) along all ξ' with $\xi \subset \xi' \subset p$.* $\square$

In light of Lemma 3.1, we define a requirement to be *active along a path* $p \in [T]$, or *satisfied along a path* $p \in [T]$, if it is active, or satisfied, respectively, along all sufficiently long $\xi \subset p$.

Construction: The construction proceeds in stages s, each of which consists of substages $t \leq s$. At substage t, a strategy $\xi \in T$ of length t is eligible to act. (So at substage 0 of any stage s, the root $\emptyset$ is eligible to act.)

We first need to fix some further notation. A strategy $\xi \in T$ is *initialized* by making all its parameters and its functionals undefined, canceling all links to or from it, canceling its restraint, and calling ξ no longer stopped (if it currently has stopped, as defined below). An $\mathcal{I}$-strategy $\xi \in T$ is *reset* by making all its parameters (except for its killing number) and all its functionals undefined, canceling all links to or from it, and initializing all strategies $> \xi$. A number is *chosen big* by choosing it above the current stage and above any number mentioned so far in the construction.

All Σ_2^0-sets A (for $a \in P$) constructed by us are Δ_2^0 (with the Δ_2^0-approximation given by our construction). By the above remark, we also force the opponents' sets Z_i to be Δ_2^0 (with the given Δ_2^0-approximation). We can thus always take all sets A and Z_i as given at the current stage.

Now the action of a strategy ξ (of length t) eligible to act at substage t of stage s depends on the type of requirement assigned to ξ:

Case 1: ξ has been assigned to $\mathcal{T}_{(Z_i)_{i\leq k},(\Psi_{i,j})_{i\leq k; j\leq k_i}}$: If s is not ξ-expansionary (as defined above) then end the substage by letting $\xi\hat{\ }\langle fin\rangle$ be eligible to act next. Otherwise, we ensure the correctness of the functional $\Gamma(\bigoplus_{l\leq m} B_l \oplus \bigoplus_{i\leq k} Z_i)$. For any $x \in \Gamma(\bigoplus_{l\leq m} B_l \oplus \bigoplus_{i\leq k} Z_i) - H$ and each axiom $\langle x, F\rangle \in \Gamma$ with $F \subseteq \bigoplus_{l\leq m} B_l \oplus \bigoplus_{i\leq k} Z_i$, we remove the number corresponding to some $y \in F$ from the appropriate set B so as to minimize the priority of any strategy $\xi' \supseteq \xi$ such that y is restrained or weakly restrained into B by ξ'. We then initialize any strategy $\geq$ any such ξ' the restraint of which has been injured, or $>$ any such ξ' the weak restraint of which has been injured. If the weak restraint of such ξ' has been injured, we also cancel the link from ξ' to all $\xi'' \subset \xi'$ the Θ-computations of which have been destroyed since the link was established.

Similarly, for any $x \in H - \Gamma(\bigoplus_{l\leq m} B_l \oplus \bigoplus_{i\leq k} Z_i)$, we distinguish two subcases:

(1) If the use $\gamma(x)$ is not defined then we choose a big number x_b, for each $b \in \mathcal{B}(y)$, and a new big use $\gamma(x) >$ all the x_b's, enumerate the axiom

$$\langle x, \bigoplus_{l\leq m}(B_m \restriction \gamma(x) \cup \{x_b\}) \oplus \bigoplus_{i\leq k}(Z_i \restriction \gamma(x))\rangle$$

into Γ and put x_b into B for each $b \in \mathcal{B}(y)$.

(2) If the use $\gamma(x)$ is defined then we enumerate the axiom

$$\langle x, \bigoplus_{l \leq m}(B_m \restriction \gamma(x)) \oplus \bigoplus_{i \leq k}(Z_i \restriction \gamma(x))\rangle$$

into Γ.

Now end the substage by letting $\xi\hat{\ }\langle\infty\rangle$ be eligible to act next.

Case 2: ξ has been assigned to $\mathcal{I}_{\Theta}^{d,c}$: Proceed as in the first applicable subcase:

Subcase 2.1: ξ has already *stopped* (as defined in Subcase 2.5.3 or Subcase 2.6.2 below): Then end the substage by letting $\xi\hat{\ }\langle fin\rangle$ be eligible to act next.

Subcase 2.2: ξ has not been assigned a *killing number* w_ξ: Then we choose a big killing number w_ξ and end the stage.

Subcase 2.3: ξ has not been assigned a *witness* n: Then we choose a big witness n, enumerate n into D, and end the stage.

So in the remaining subcases, we can assume that ξ's killing number w_ξ and witness n are defined. We have to distinguish subcases by the position of c within $\mathcal{P}$.

Subcase 2.4: $c \geq \mathcal{B}(y)$ but $c \not\geq \mathcal{Z}(y)$, and there is some $\mathcal{T}$-requirement active along ξ: Set $i^* = i_c$, and fix $j^* \leq k_{i^*}$ least with $a_{i^*,j^*} \not\geq \mathcal{B}(y)$. Let $\eta \subset \xi$ be the longest $\mathcal{T}$-strategy such that its $\mathcal{T}$-requirement is active along ξ.

First, check whether some $\mathcal{I}$-strategy $\hat{\xi} \supseteq \xi\hat{\ }\langle\infty\rangle$ has linked to ξ (and this link has not been canceled). (Note that, by our construction, there can be at most one such $\hat{\xi}$.) Proceed as in the first applicable subcase.

Subcase 2.4.1: Some set A (for $a \in \mathcal{P}$) has changed on a number $< w_\xi$: Then reset ξ and end the stage.

Subcase 2.4.2: ξ's current witness n was chosen by an $\mathcal{I}$-strategy $\xi' \supseteq \xi\hat{\ }\langle\infty\rangle$ (as defined in Subcase 2.5.3), and ξ' has been initialized since: Then make ξ's witness undefined and end the stage.

Subcase 2.4.3: There is such a link and $n \in \Theta(C) - D$: Then end the substage by letting $\xi\hat{\ }\langle fin\rangle$ be eligible to act next.

Subcase 2.4.4: There is such a link but ξ's computation $\Theta(C; n)$ from the time the link from $\hat{\xi}$ was created has been destroyed: Then $\hat{\xi}$ *weakly restrains* all numbers currently in C and *restrains* all numbers currently in A_{i^*,j^*}, we cancel the link from $\hat{\xi}$ to ξ, and end the stage.

Subcase 2.4.5: For some axiom $\langle x, F\rangle$ in Δ with $F \subseteq C$, x has not been in Z_{i_c} at some stage since this axiom was enumerated into Δ: For each such axiom, we remove a number $y \in F$ from C so as to minimize the priority of any strategy $\xi' \supseteq \xi$ such that y is restrained or weakly restrained into C by ξ'. We then initialize any strategy $\geq$ any such ξ' the restraint of which has been injured, or $>$ any such ξ' the weak restraint of which has been injured. If the weak restraint of such ξ' has been injured, we also cancel the link from ξ' to all $\xi'' \subset \xi'$ the Θ-computations of which have been destroyed since the link was established. Now end the stage.

Subcase 2.4.6: There is no such link and $n \in \Theta(C) - D$: Then end the substage by letting $\xi\hat{\ }\langle fin\rangle$ be eligible to act next.

Subcase 2.4.7: $n \in D - \Theta(C)$: Then end the substage by letting $\xi\hat{\ }\langle fin\rangle$ be eligible to act next.

Subcase 2.4.8: $n \in \Theta(C) \cap D$: Remove n from D and end the stage.

Subcase 2.4.9: $n \notin D \cup \Theta(C)$: Then for each axiom $\langle x, G\rangle$ in η's functional $\Gamma(\bigoplus_{l \leq m} B_l \oplus \bigoplus_{i \leq k} Z_i)$ (for η as defined at the beginning of Subcase 2.4) such that $x \geq w_\xi$ and such that $G \subseteq \bigoplus_{l \leq m} B_l \oplus \bigoplus_{i \leq k} Z_i$, remove the number corresponding to $y \in G$ from the appropriate set B so as to minimize the priority of any strategy $\xi' \supseteq \xi$ such that y is restrained or weakly restrained by ξ'. We also make the use $\gamma(x)$ undefined for all such x. Similarly, for each axiom $\langle x, G\rangle$ in some $\Delta^*(C^*)$ built by some $\mathcal{I}^*$-strategy ξ^* with $\eta \subset \xi^* \subset \xi$ such that $x \geq w_\xi$ and such that $G \subseteq C^*$, remove a number $y \in G$ from the set C^* so as to minimize the priority of any strategy $\xi' \supseteq \xi$ such that y is restrained or weakly restrained by ξ'. We also make the use $\delta^*(x)$ undefined for all such x. We then initialize any strategy $\geq$ any such ξ' the restraint of which has been injured, or $>$ any such ξ' the weak restraint of which has been injured. If the weak restraint of such ξ' has been injured, we also cancel the link from ξ' to all $\xi'' \subset \xi'$ the Θ-computations of which have been destroyed since the link was established.

Furthermore, for any $x \in Z_{i_c} - \Delta(C)$, we distinguish two subcases:

(1) If the use $\delta(x)$ is not defined then we choose a big number x_c and a big use $\delta(x) > x_c$, enumerate the axiom $\langle x, C \restriction \delta(x) \cup \{x_c\}\rangle$ into Δ, and put x_c into C.

(2) If the use $\delta(x)$ is defined then we enumerate the axiom $\langle x, C \restriction \delta(x)\rangle$ into Δ.

Finally, end the substage by making ξ's witness n undefined and letting $\xi\hat{\ }\langle\infty\rangle$ be eligible to act next. This ends the description of Subcase 2.4.

Subcase 2.5: $c \geq \mathcal{B}(y)$ but $c \not\geq \mathcal{Z}(y)$, and there is no $\mathcal{T}$-requirement active along ξ: Set $i^* = i_c$, and fix $j^* \leq k_{i^*}$ least with $a_{i^*,j^*} \not\geq \mathcal{B}(y)$. Proceed as in the first applicable subcase.

Subcase 2.5.1: Some set A (for $a \in \mathcal{P}$) has changed on a number $< w_\xi$: Then reset ξ and end the stage.

Subcase 2.5.2: $n \in D - \Theta(C)$: Then end the substage by letting $\xi\hat{\ }\langle \mathit{fin}\rangle$ be eligible to act next.

Subcase 2.5.3: $n \in D \cap \Theta(C)$: Remove n from D and say ξ *stops*. Let $\xi^0 \subset \cdots \subset \xi^{q-1}$ be all the $\mathcal{I}$-strategies ξ^p with $i_{c^p} = i^*$ and $\xi^p\hat{\ }\langle\infty\rangle \subseteq \xi$ which destroy some functional Γ^p built by $\eta^p \subset \xi^p$, say. (Set $\xi^q = \xi$ and allow $q = 0$, in which case we end the stage immediately.) Now, for each $p = 0, \ldots, q-1$, create a link from ξ to ξ^p, request that ξ^p restore its computation $\Theta^p(C^p; n^p)$ (for the most recent witness n^p for which ξ^p has found a computation, which we now again declare to be ξ^p's witness) by restoring B for all $b \in \mathcal{B}(y)$ as needed, *weakly restrain* all numbers currently in C, *restrain* all numbers currently in A_{i^*,j^*}, and do not redefine, extend or correct Δ_p until $\Theta^p(C^p; n^p)$ is destroyed (as detailed in Case 2.4), and end the stage.

Subcase 2.6: $c \not\geq \mathcal{B}(y)$ or $c \geq \mathcal{B}(y) \cup \mathcal{Z}(y)$:

Subcase 2.6.1: $n \in D - \Theta(C)$: Then end the substage by letting $\xi\hat{\ }\langle \mathit{fin}\rangle$ be eligible to act next.

Subcase 2.6.2: $n \in D \cap \Theta(C)$: Remove n from D, say ξ *stops*, restrain all numbers currently in C, and end the stage.

Case 3: ξ has been assigned to $\mathcal{L}^a_{\Omega,x}$: If $x \notin \Omega(A)$ or if some $\mathcal{L}^a_{\Omega,x}$-strategy $\xi' \leq \xi$ has already stopped (as defined below), then end the substage by letting $\xi\hat{\ }\langle \mathit{fin}\rangle$ be eligible to act next. Otherwise, let $\xi' \leq \xi$ be the highest-priority $\mathcal{L}^a_{\Omega,x}$-strategy that has ever been eligible to act, let ξ' restrain all numbers currently in A, say that ξ' has *stopped*, initialize all strategies $> \xi'$, and end the stage.

End of Stage s: Initialize all strategies $\geq \xi\hat{\ }\langle \mathit{fin}\rangle$ where ξ is the longest strategy having been eligible to act at stage s.

Verification: We will now verify in a sequence of lemmas that the above construction proves that condition (2) of Theorem 1.5 ensures non-extendibility.

We first define the true path of the construction $f \in [T]$ by induction on the length. Given $\xi = f \restriction n$, we set

$$f(n) = \begin{cases} \infty & \text{if } \xi\hat{\ }\langle\infty\rangle \in T \text{ and is eligible to act infinitely often;} \\ \mathit{fin} & \text{otherwise.} \end{cases}$$

Lemma 3.2. *All sets A (for $a \in P$) are Δ^0_2, with the Δ^0_2-approximation as given.*

Proof. A set A changes at a number x for one of three reasons: (i) x is a witness for diagonalization, which is enumerated (in Subcase 2.3) or extracted (in Subcase 2.4.8, 2.5.3, or 2.6.2). (ii) x is a use number w_a for some Γ or Δ, which is enumerated (in Case 1 or Subcase 2.4.9, respectively) or extracted for correction (in Case 1 or Subcase 2.4.5, respectively) or for killing (in Subcase 2.4.9). (iii) x is a number which is enumerated into A as part of restoring a computation (in Subcase 2.5.3). Obviously, (i) and (ii) can apply for enumeration of x at most once each. And (iii) can apply for enumeration of x only by strategies with a killing number $\leq x$, so there are only finitely many such strategies. Let ξ be the highest-priority such strategy. If ξ enumerates x into A after all enumeration of x under (i) and (ii) has stopped, then all strategies $> \xi$ are initialized and so can never want to restore computations using x. If ξ's computation using x is later destroyed (and x is extracted), then ξ will later use a different computation not involving x, and so x is permanently out of A. □

Lemma 3.3. *(i) Every strategy along the true path is eligible to act infinitely often.*

(ii) The restraint of every strategy along the true path is injured at most finitely often. The weak restraint of every strategy $\leq$ the true path is injured at most finitely often. No strategy along the true path is initialized or reset more than finitely often.

Proof. We proceed by induction on the length of $\xi \subset f$.

(i) This is trivial if $\xi = \emptyset$, so assume $|\xi| > 0$. By the definition of f, the claim is again trivial if $\xi = \xi^-\hat{\ }\langle\infty\rangle$, so assume $\xi = \xi^-\hat{\ }\langle \mathit{fin}\rangle$. If ξ^- is a $\mathcal{T}$- or $\mathcal{L}$-strategy, then, by (ii) for ξ^-, ξ^- ends the stage at most finitely often, so (i) for ξ follows. So assume that ξ^- is an $\mathcal{I}$-strategy and, for the sake of a contradiction, that ξ^- ends the stage at cofinitely many stages at which it is eligible to act, say, always after stage s_0, and that ξ^- is also not initialized or reset after stage s_0. Again, Subcases 2.5.3 and 2.6.2 can apply at most finitely often; so Subcases 2.2, 2.3, 2.4.1, 2.4.2, 2.4.5, or 2.4.8 must apply cofinitely often. Clearly, Subcases 2.2, 2.3, 2.4.2, and 2.4.8 can apply at most finitely often after stage s_0. Subcase 2.4.5 can apply at most finitely often since no new Δ-axioms can be defined after stage s_0. Finally, Subcase 2.4.1 can apply at most finitely often after stage s_0 since w_{ξ^-} is fixed and all $A \restriction w_{\xi^-}$ eventually settles down by Lemma 3.2.

(ii) Fix a stage s_0 such that ξ^- is not initialized and such that no strategy $<_L \xi$ is eligible to act after stage s_0. Then the restraint of any $\xi' \leq \xi^-$, and the weak restraint of any $\xi' < \xi^-$, cannot be injured after stage s_0. (Set $s_0 = 0$ if $\xi = \emptyset$.)

So after stage s_0, ξ can be initialized only if (1) ξ^- ends the stage and $\xi^-\hat{\ }\langle \mathit{fin}\rangle = \xi$; or if (2) the weak restraint of some strategy $\xi' < \xi$ is injured via Case 1, or via Subcase 2.4.5 or 2.4.9; or (3) the restraint of some strategy $\xi' \leq \xi$ is injured via Case 1, or via Subcase 2.4.5 or 2.4.9.

For (1) to apply, ξ^- must end the stage. But this cannot happen infinitely often unless $\xi^-\hat{\ }\langle\infty\rangle \subset f$ since ξ^- cannot end the stage infinitely often without also taking the outcome ∞ infinitely often, a contradiction.

For (2) to apply, by our inductive assumption, any such strategy ξ' must satisfy $\xi' \geq \xi^-$ and so $\xi' = \xi^-$. But then ξ^- must be an $\mathcal{I}$-strategy for which Subcase 2.5 applies, and the link from ξ^- to some $\xi^p \subseteq \xi^-$ (for ξ^p as defined in Subcase 2.5.3 for ξ^-) is canceled when ξ is initialized at stage $s > s_0$, say. We first observe that the weak restraint of ξ^- cannot be injured by Γ- or Δ-correction for some Γ or Δ not active along ξ^p (since this destruction will have happened just before ξ^p found its current computation). Furthermore, the computation of ξ^p cannot be injured by Δ-correction by some $\xi^{p'}$ (the Δ of which is still active along ξ^p) since when the link from ξ^- to ξ^p was created at stage $s' < s$, say, a link was also created from ξ^- to $\xi^{p'}$, and this link was canceled at a stage $s'' \leq s$ when the computation of $\xi^{p'}$ was destroyed. By induction on p, we may assume that this computation was destroyed by Γ-correction by $\eta^{p'}$, say. But then this Γ-correction was necessitated by the restoration of all of the Z_i's of $\eta^{p'}$, and so, by the $A_{i,j}$-restraint imposed by ξ^- at stage s'', Δ-correction by $\xi^{p'}$ becomes unnecessary.

We have thus shown that for (2) to apply after stage s_0, the weak restraint of ξ^- must be injured by Γ-correction, destroying a computation of some $\xi^p \subset \xi^-$ along which this Γ is still active. But every time this happens, a link from ξ^- is canceled and not created again later, so this can happen at most finitely often. (In particular, the computation of ξ^- itself can never be destroyed since there are no active Γ along ξ^-.)

For (3) to apply, by our inductive assumption, any such strategy ξ' must satisfy $\xi' > \xi^-$ and so $\xi' = \xi$. But then ξ must be an $\mathcal{I}$-strategy for which Subcase 2.5 or 2.6 applies, or an $\mathcal{L}^a$-strategy. The restraint of an $\mathcal{L}^a$-strategy applies only to A for $a \not\geq \mathcal{B}(y)$, and this restraint is eventually the highest-priority one possibly being injured via Case 1, or via Subcase 2.4.5 or 2.4.9. But this injury is due to Γ- or Δ-correction with some oracle $\geq \mathcal{B}(y)$, so Δ-correction will never injure, and injury due to Γ-correction can be eventually be avoided since we try to minimize the priority of the injured strategy by a careful choice of the set B via which Γ is corrected. The same holds true for the restraint of an $\mathcal{I}^{c,d}$-strategy if $c \not\geq \mathcal{B}(y)$, so assume that $c \geq \mathcal{B}(y)$. We are thus left with two possibilities: (3a) Subcase 2.5 applies to ξ, or (3b) Subcase 2.6 applies to ξ and $c \geq \mathcal{B}(y) \cup \mathcal{Z}(y)$.

For (3a) to apply, the injury cannot be due to Γ-correction since Γ will just have been destroyed at the stage at which ξ imposes its restraint. Also, if the injury is due to Δ-correction (where Δ computes some set Z_i from some oracle C^*, say) then $a \not\geq z_i$ (else Z_i is restrained by ξ via C by our definition of expansionary stages) and $c \geq c^*$. Eventually, Δ-correction injuring ξ can only be performed by $\mathcal{I}$-strategies $\subset \xi$, so let ξ_0 be the highest-priority such injuring ξ infinitely often and note that $\xi_0\hat{\ }\langle\infty\rangle \subseteq \xi$ and that some $\mathcal{T}$-requirement is satisfied along ξ via ξ_0 (else Δ will just have been destroyed at the stage at which ξ imposes its restraint). Say, ξ_0 performs its Δ-correction via Subcase 2.4.5 at stage s since some z, which was in Z_i at the stage $s' < s$ at which the Δ-axiom was defined, is no longer in Z_i. This z can have left Z_i only between stages s' and s only because ξ itself extracted a witness n, say (else ξ would have been initialized already upon the extraction of n). But then ξ, while extracting n, also restores the Θ_0-computation of ξ_0, and ξ_0 cannot correct Δ until its Θ_0-computation is again destroyed. This destruction means that ξ's weak restraint for ξ_0 was injured.

For (3b) to apply, the injury can again not be due to Γ-correction, but this time since Γ will just have been corrected at the stage at which ξ imposes its restraint and since $h \leq c$ by our assumption on h, so ξ restrains also the set H. The argument for Δ-correction is the same as for (3a). $\square$

Lemma 3.4. *Each $\mathcal{C}$-requirement is satisfied.*

Proof. The satisfaction of the requirements $\mathcal{C}^0$ and $\mathcal{C}^1$ follows immediately by the definition of C for $c \in \{0,1\}$. The satisfaction of $\mathcal{C}^{d,c}$ for each pair $d < c$ in $P - \{0,1\}$ follows as outlined above: A number x ever entering D must be targeted for a set A (with $a \leq d$) by stage x, and then $x \in D$ iff $x \in C$. $\square$

Lemma 3.5. *(i) Every $\mathcal{L}^a_{\Omega,x}$-requirement is satisfied.*

(ii) For every $a \in P$ with $a \not\geq \mathcal{B}(y)$, A is a low set.

(iii) Every set Z_i (used by a $\mathcal{T}$-strategy $\xi \subset f$, for some $i \leq k$) is Δ^0_2, with the Δ^0_2-approximation as given.

Proof. (i) Suppose that for infinitely many stages s, $x \in \Omega(A)[s]$. Let $\xi \subset f$ be the $\mathcal{L}^a_{\Omega,x}$-strategy via which $\mathcal{L}^a_{\Omega,x}$ is satisfied along f, and fix a stage s_0 at which ξ is eligible to act and such that no strategy $\leq \xi$ is initialized after stage s_0, as well as a stage $s > s_0$ such that $x \in \Omega(A)[s]$. Then at stage s, if no $\mathcal{L}^a_{\Omega,x}$-strategy $\xi' \leq \xi$ has already stopped, this will happen at stage s. Since ξ' is not initialized after stage s_0, its A-restraint is never injured, and so $x \in \Omega(A)$ as desired.

(ii) and (iii) are now immediate by (i). $\square$

Lemma 3.6. *(i) If a requirement $\mathcal{T}_{(Z_i)_{i\leq k},(\Psi_{i,j})_{i\leq k; j\leq k_i}}$ is active along the true path f (via a $T_{(Z_i)_{i\leq k},(\Psi_{i,j})_{i\leq k; j\leq k_i}}$-strategy $\xi \subset f$) then the set H is correctly computed by ξ's functional $\Gamma(\bigoplus_{l\leq m} B_l \oplus \bigoplus_{i\leq k} Z_i)$.*

(ii) If a requirement $\mathcal{T}_{(Z_i)_{i\leq k},(\Psi_{i,j})_{i\leq k; j\leq k_i}}$ is satisfied along the true path f (via a $T_{(Z_i)_{i\leq k},(\Psi_{i,j})_{i\leq k; j\leq k_i}}$-strategy $\xi \subset f$) then for some $i \leq k$ and some $j, j' \leq k_i$, $\Psi_{i,j}(A_{i,j}) \neq \Psi_{i,j}(A_{i,j'})$.

(iii) If a requirement $\mathcal{T}_{(Z_i)_{i\leq k},(\Psi_{i,j})_{i\leq k; j\leq k_i}}$ is satisfied along the true path f (via a $\mathcal{I}^{c,d}$-strategy $\xi \subset f$) then ξ's functional $\Delta(C)$ correctly computes the set D.

Proof. (i) Fix an argument x of $\Gamma(\bigoplus_{l\leq m} B_l \oplus \bigoplus_{i\leq k} Z_i)$. By Γ-correction, if $x \notin H$ then $x \notin \Gamma(\bigoplus_{l\leq m} B_l \oplus \bigoplus_{i\leq k} Z_i)$. So suppose that $x \in H$ and that any $\mathcal{I}$-strategy $\xi' \supseteq \xi\hat{\ }\langle\infty\rangle$ along which $T_{(Z_i)_{i\leq k},(\Psi_{i,j})_{i\leq k; j\leq k_i}}$ is active via ξ and for which the killing number $w_{\xi'}$ is $\leq x$ has stopped killing Γ (under Subcase 2.4.9 of the construction). By the definition of the use $\gamma(x)$, this use must eventually settle down. By Lemma 3.5 (iii), each Z_i in the oracle of Γ is a Δ^0_2-set (under the given approximation), and so is each set B_l in the oracle of Γ by Lemma 3.2. Thus ξ will eventually enumerate a Γ-axiom which ensures that $x \in \Gamma(\bigoplus_{l\leq m} B_l \oplus \bigoplus_{i\leq k} Z_i)$.

(ii) In this case, there are only finitely many ξ-expansionary stages, so eventually for some fixed $i \leq k$ and some fixed x, either

$$x \in Z_i \text{ and } x \notin \bigcap_{j\leq k_i} \Psi_{i,j}(A_{i,j}),$$

or

$$x \notin Z_i \text{ and } x \in \bigcap_{j\leq k_i} \Psi_{i,j}(A_{i,j}).$$

Either way, we have that $Z_i \neq \Psi_{i,j}(A_{i,j})$ for some $i \leq k$, as claimed.

(iii) Fix an argument x of $\Delta(C)$. By Δ-correction (under Subcase 2.4.5), if $x \notin Z_i$ then $x \notin \Delta(C)$. So suppose that $x \in Z_i$ and that any $\mathcal{I}$-strategy $\xi' \supseteq \xi\hat{\ }\langle\infty\rangle$ along which $T_{(Z_i)_{i\le k},(\Psi_{i,j})_{i\le k;j\le k_i}}$ is satisfied via ξ and for which the killing number $w_{\xi'}$ is $\le x$ has stopped killing Δ (under Subcase 2.4.9 of the construction). By the definition of the use $\delta(x)$, this use must eventually settle down. Now the oracle C of Δ is a Δ_2^0-set by Lemma 3.2. Thus ξ will eventually enumerate a Δ-axiom which ensures $x \in \Delta(C)$. □

Lemma 3.7. *Every requirement $\mathcal{I}_\Theta^{c,d}$ is satisfied.*

Proof. Fix the $\mathcal{I}_\Theta^{c,d}$-strategy $\xi \subset f$ via which $\mathcal{I}_\Theta^{c,d}$ is satisfied along f, and fix a stage s_0 at which ξ is eligible to act such that no strategy $\le \xi\hat{\ }\langle \mathit{fin}\rangle$ is initialized or reset after stage s_0. Then after stage s_0, ξ has a fixed witness n, which is initially enumerated into D. By our assumption on stage s_0, any restraint ξ imposed after stage s_0 is never injured. We can now distinguish the following three cases:

Case I: ξ never finds a computation $\Theta(C;n)$ after stage s_0: Then $n \in D$, and $\mathcal{I}_\Theta^{c,d}$ is satisfied.

Case II: ξ finds a computation $\Theta(C;n)$ after stage s_0: Then this computation cannot be destroyed later since either ξ is able to restrain C (via Subcase 2.5 or 2.6 of the construction), or since ξ would, after the destruction of the computation, take outcome ∞ (via Subcase 2.4.9 of the construction), contrary to the choice of s_0. Also, ξ removes n from D, and so $\mathcal{I}_\Theta^{c,d}$ is satisfied.

Case III: ξ has a computation $\Theta(C;n)$ which is restored by an $\mathcal{I}$-strategy $\xi' \supseteq \xi$ via a link from ξ' to ξ existing at stage s_0: Then this computation cannot be destroyed later since ξ would, after the destruction of the computation, end the stage and initialize $\xi\hat{\ }\langle \mathit{fin}\rangle$ (via Subcase 2.4.2 of the construction), contrary to the choice of s_0. Also ξ will have removed n from D, and again $\mathcal{I}_\Theta^{c,d}$ is satisfied.

Note that this exhausts all possibilities since if there is a link to ξ at stage s_0, then Case III must apply, and otherwise no such link can be created after stage s_0 by our assumption on stage s_0. □

The fact that condition (2) of Theorem 1.5 ensures non-extendibility now follows by Lemma 3.1 and Lemmas 3.5–3.7.

4. The third non-extendibility theorem. In this section, we show that condition (3) of Theorem 1.5 ensures non-extendibility. Fix x and y as in condition (3). We may assume that condition (1) of Theorem 1.5 fails for x and y, i.e., that $\mathcal{BA}(x)) \not\le \mathcal{AB}(y)$, in particular that $\mathcal{BA}(x) - \mathcal{B}(y) \neq \emptyset$.

We will build an embedding of $\mathcal{P}$ into the Σ_2^0-enumeration degrees. (In this section, we will assume the convention that an element of $\mathcal{P}$ denoted by a lower-case letter is mapped to the Σ_2^0-enumeration degree denoted by the corresponding boldface letter, which is represented by the Σ_2^0-set denoted by the corresponding upper-case letter.)

We will show that our embedding of $\mathcal{P}$ cannot even be extended to an embedding of $\mathcal{P} \cup \{x, y\}$ (allowing $y \in \mathcal{P}$) meeting only the conditions

$$x \not\le y,$$
$$\mathcal{B}(x) \subseteq \mathcal{B}(y), \text{ and}$$
$$\mathcal{B}(y) \not\subseteq \mathcal{A}(x)$$

So we may assume without loss of generality that

$$\mathcal{Q} = \mathcal{P} \cup \{x, y\} \text{ (allowing } y \in \mathcal{P}\text{), and}$$
$$\mathbf{y} = \bigcup_{b \in \mathcal{B}(y)} \mathbf{b}.$$

In our construction, this allows us to build the set Y ourselves even in the case when $y \notin \mathcal{P}$.

We can ensure that the embedding preserves 0 and 1 by mapping 0 to $\mathbf{0}_e = \deg_e(\emptyset)$, and 1 to $\mathbf{0}'_e = \deg_e(\overline{K})$, respectively. Note the below requirements will cause numbers to be targeted into the image of 0. And any embedding not necessarily mapping 1 to $\mathbf{0}'_e$ can be modified by changing the image of 1 to be $\mathbf{0}'_e$. The new embedding will still satisfy all the requirements since for the $\mathcal{S}$-requirements below, $\mathcal{A}(x)$ cannot be $\{1\}$.

We set $\mathcal{C} = \mathcal{BA}(x) - \mathcal{B}(y)$. We also fix an element $b_0 \in \mathcal{B}(y) - \mathcal{BA}(x)$. (Both $\mathcal{BA}(x) - \mathcal{B}(y)$ and $\mathcal{B}(y) - \mathcal{BA}(x)$ are non-empty by the first paragraph of this section and by the third condition above, respectively.)

We now need to meet, for all $c, d \in \mathcal{P}$, all $e \in \mathcal{P} \cup \{y\}$, all Σ^0_2-sets X, all $|\mathcal{A}(x)|$-tuples $\vec{\Phi}$ of enumeration operators, and all enumeration operators Ψ, the following

Requirements:

$$\begin{aligned}
\mathcal{P}: &\quad Y \leq_e \bigoplus_{b \in \mathcal{B}(y)} B \text{ (as discussed above)},\\
\mathcal{Q}^{c,e}: &\quad C = \Theta_{c,e}(E) \text{ (if } c < e\text{)},\\
\mathcal{S}_{X,\vec{\Phi}}: &\quad \forall a \in \mathcal{A}(x)\, (X = \Phi_a(A)) \implies\\
&\qquad \exists \Gamma\, (X = \Gamma(Y)) \text{ or } \exists c \in \mathcal{C}\, \exists \Delta\, (C = \Delta(X)), \text{ and}\\
\mathcal{T}^{c,d}_{\Psi}: &\quad C \neq \Psi(D) \text{ (if } c \not\leq d\text{)},
\end{aligned}$$

where the Γ's, Δ's, and Θ's are enumeration operators built by us.

Strategy for $\mathcal{P}$: Any number z' entering or leaving Y will turn out to be the $\Theta_{b,y}$-use of a number z targeted for some set B with $b \in \mathcal{B}(y)$ (i.e., we have a $\Theta_{b,y}$-axiom $\langle z, \{z'\}\rangle$), and once this $\Theta_{b,y}$-axiom is enumerated, we will ensure that $z' \in Y$ iff $z \in B$. This will clearly ensure $\mathcal{P}$.

Strategy for $\mathcal{Q}^{c,e}$: We will explicitly build enumeration operators $\Theta_{c,e}$ to ensure the satisfaction of $\mathcal{Q}^{c,e}$. Each such enumeration operator will be a so-called s-operator, i.e., all axioms of $\Theta_{c,e}$ will be of the form $\langle z, \{z'\}\rangle$ for some numbers z and z' (or of the form $\langle z, \emptyset\rangle$). In that case, we call any such z' a $\Theta_{c,e}$-*use* (or simply Θ-*use*) of z.

We pause briefly to remark on the overall setup of targeting numbers and picking Θ-uses. A number z will first be picked either as

(i) a diagonalization witness for $\mathcal{T}^{c,d}$, or as
(ii) a "killing point" for a functional $\Gamma(Y)$ (used by a $\mathcal{T}$-strategy to destroy Γ, to be replaced by a functional $\Delta(X)$), or as
(iii) a "coding number" for $\Gamma(Y; z')$ (i.e., z will be a number in the oracle set F of a Γ-axiom $\langle z', F\rangle$), or as
(iv) a Θ-use

As we go down the tree of strategies, the set of numbers that may enter or leave any set C will be restricted to an infinite "stream" S of numbers. This stream will be generated from numbers used at earlier stages which now have to be reused. Whenever we target a number for a set C_0, we will be restricted to numbers from the stream S, with $\Theta_{c_0,c}$-uses also from the stream S (where $c_0 < c$). We will have to ensure that there are always infinitely many "independent" numbers available for each set C, in particular, numbers which are not yet Θ-uses of any number which has not yet been dumped, and all of whose current Θ-uses have been removed.

Returning to the discussion of satisfying $\mathcal{Q}^{c,e}$, we now proceed as follows:

When a number z is first targeted for a set C as a witness, killing point, or coding number, then we pick not only an "independent" number $z = z_c$ from the current stream S but also possibly distinct numbers z_e from the current stream S (for each $e \in \mathcal{P} \cup \{y\}$ with $e > c$), and we add an axiom $\langle z_d, \{z_e\}\rangle$ into $\Theta_{d,e}$ for each $d \in \mathcal{P}$ and $e \in \mathcal{P} \cup \{y\}$ with $c \leq d < e$. These numbers z_e (for $e > c$) will not appear in the stream of subsequent strategies since they are no longer "independent"; z_c, however, can enter another stream, at which point the z_e's are removed.

If z is a killing point or a coding number for some Γ, then z and all its Θ-uses will be *dumped* into their respective target sets when Γ is canceled, and none of these numbers will be reused. Similarly, if z is a diagonalization witness then z and all its Θ-uses will be *dumped* into their respective target sets when the strategy that picked z is initialized. On the other hand, if z is a diagonalization witness for a $\mathcal{T}^{c,d}$-strategy α (where $c \leq \mathcal{A}(x)$) which turns out not to be "Γ-cleared" for some higher-priority enumeration operator Γ, then, for each $e > c$, z's current $\Theta_{c,e}$-use z_e will be extracted from its respective target set E as well as from the stream S. Now z can be retargeted for C with different $\Theta_{c,e}$-uses since the previous Θ-axioms cannot apply.

Of course, when a number z is dumped into a set C then we add an axiom $\langle z, \emptyset\rangle$ into $\Theta_{c,e}$ for all $e \in \mathcal{P} \cup \{y\}$ with $e > c$.

Strategy for $\mathcal{S}$: This strategy will build an enumeration operator $\Gamma(Y)$ trying to compute X. We first define a stage s to be *expansionary* if $s = 0$ or

$$\bigcup_{a \in \mathcal{A}(x)} \Phi_a(A)[s_0] \subseteq \bigcap_{a \in \mathcal{A}(x)} \Phi_a(A)[s]$$

where s_0 denotes (the end of) the most recent expansionary stage $< s$. (The idea here is that we cannot measure the length of agreement between the Σ_2^0-sets $\Phi_a(A)$ as is usually done for computably enumerable or Δ_2^0-sets; instead, we merely measure whether any number z which was in the union of the $\Phi_a(A)$ at the end of stage s_0 (and which thus must still be in this union at stage s by the implicit restraint imposed between stages s_0 and s) must now be in the intersection of the $\Phi_a(A)$; otherwise, we do not believe that all the $\Phi_a(A)$ coincide.)

At non-expansionary stages, the $\mathcal{S}$-strategy now does nothing and simply takes the finite outcome. At an expansionary stage, the $\mathcal{S}$-strategy will take the infinite outcome, and it acts as follows: For the least z currently in $\bigcap_{a \in \mathcal{A}(x)} \Phi_a(A) - \Gamma(Y)$ (if any) and for each $b \in \mathcal{B}(y)$, the $\mathcal{S}$-strategy picks an available coding number z_b (targeted for B), enumerates z_b into B, and adds an axiom $\langle z, F\rangle$ into Γ where the oracle set F is the set containing the current $\Theta_{b,y}$-uses of all z_b's as well the current $\Theta_{b_0,y}$-uses of all current "killing points" for Γ defined by $\mathcal{T}$-strategies below the $\mathcal{S}$-strategy. (Recall that b_0 is our fixed element in $\in \mathcal{B}(y) - \mathcal{B}\mathcal{A}(x)$. Since all these killing points will be in Y whenever the $\mathcal{S}$-strategy is eligible to act, these killing points will not affect Γ from the $\mathcal{S}$-strategy's point of view. The purpose of the killing points, as explained in more detail below, is to allow a $\mathcal{T}$-strategy below the $\mathcal{S}$-strategy to make $\Gamma(Y)$ finite by removing its killing point from Y at infinitely many stages, each time for only one stage.) Furthermore, for any z currently in $\Gamma(Y) - \bigcup_{a \in \mathcal{A}(x)} \Phi_a(A)$, the $\mathcal{S}$-strategy removes z_{b_0} from B_0.

Strategy for $\mathcal{T}$ in isolation: This strategy is merely the Friedberg-Muchnik strategy for Σ^0_2-enumeration degrees: We fix a witness z at which we wish to diagonalize C against $\Psi(D)$, add z to C, and wait until (if ever) $z \in \Psi(D)$ via some Ψ-axiom $\langle z, F \rangle$ with $F \subseteq D$. Then we remove z from C and ensure that $F \subseteq D$ from now on. This clearly ensures diagonalization and is compatible with the $\mathcal{P}$- and $\mathcal{Q}$-strategies.

Strategy for $\mathcal{T}$ below one $\mathcal{S}$-strategy: Fix a requirement $\mathcal{T}^{c,d}_{\Psi}$. If $c \not\leq \mathcal{A}(x)$ then we can change C while holding $X = \Phi_a(A)$ for some $a \in \mathcal{A}(x)$ with $a \not\geq c$, preventing $\Gamma(Y)$ from needing to be corrected. If $d \not\geq \mathcal{B}(y)$ then the $\mathcal{S}$-strategy can correct Γ by extracting the witness's coding number from B (for some $b \in \mathcal{B}(y)$ with $b \not\leq d$), and thus its $\Theta_{b,y}$-use from Y, without affecting the $\mathcal{T}$-strategy. In both these cases, the $\mathcal{T}$-strategy can thus diagonalize finitarily.

On the other hand, assume $c \leq \mathcal{A}(x)$ and $d \geq \mathcal{B}(y)$ (and so $c \in \mathcal{C}$). Then the $\mathcal{T}$-strategy will try to find a number $z \in \Psi(D)$ which can be removed from C while still maintaining $\Gamma(Y) \subseteq X$ and $z \in \Psi(D)$; if the $\mathcal{T}$-strategy ever finds such a number then it will diagonalize and stop. Otherwise, it will generate a sequence of z's such that the removal of each z causes numbers to leave X. We now restrict future C-changes to these z's. This will allow us to compute C from X via an enumeration operator Δ, which we define, thus meeting the $\mathcal{S}$-requirement via the second alternative.

More precisely, the $\mathcal{T}$-strategy proceeds as follows:

(1) Pick a fresh "killing point" k for Γ. Put k into B_0 (where, again, b_0 was chosen above once and for all as a fixed element of $\mathcal{B}(y) - \mathcal{BA}(x)$). Require all future Γ-axioms $\langle z', F' \rangle$ to include the $\Theta_{b_0,y}$-use of k in their oracle set F'.
(2) Pick a fresh witness z and put z into C.
(3) Wait for $z \in \Psi(D)$ via some axiom $\langle z, F \rangle$ at a stage s, say.
(4) Extract z from C (without restraining D for now, i.e., allowing the $\mathcal{S}$-strategy to correct Γ, thus possibly injuring $\Psi(D; z)$ for now).
(5) From now on, if ever $\Gamma(Y)[s] \subseteq \bigcup_{a \in \mathcal{A}(x)} \Phi_a(A)$ (while $z \notin C$), then cancel all action between stage s and now, restrain $F \subseteq D$, and stop. (In this case, we call the computation $\Psi(D; z)$ Γ-*cleared*.)
(6) While waiting for Step (5) to apply, put z into the stream S; restrict all future changes in $C \restriction s$ to numbers in S; extract k from B_0 for the remainder of stage s; add an axiom $\langle z, \Gamma(Y)[s] \rangle$ into Δ; add axioms $\langle z', \emptyset \rangle$ to Δ for all $z' < s$ with $z' \in C_s - S$; and restart at Step (2) with a fresh z.

The possible *outcomes* of the above $\mathcal{T}$-strategy are as follows:

(A) Waiting forever at Step (3): Then $z \in C - \Psi(D)$, and Γ is not affected since $k \in B_0$ and so its $\Theta_{b_0,y}$-use is in Y.
(B) Stopping eventually at Step (5): Then $z \in \Psi(D) - C$, and Γ is not affected since $k \in B_0$ and so its $\Theta_{b_0,y}$-use is in Y.
(C) Looping between Step (2) to Step (6) infinitely often: Then the $\mathcal{T}$-require-

ment may not be satisfied by the action of this strategy, and $\Gamma(Y)$ will be finite since $k \notin B_0$ (and thus its $\Theta_{b_0,y}$-use is not in Y) while all but finitely many Γ-axioms $\langle z', F' \rangle$ contain $\Theta_{b_0,y}$-use of k in their oracle set F'. On the other hand, $C(z) = \Delta(X; z)$ can be seen to hold for all z as follows: It is clear for all $z \notin S$ by the C-restraint in Step (6), so assume that $z \in S$. If $z \in C$ (enumerated into S at stage s, say) then $X_s \subseteq X$ (assuming here that no other strategies remove numbers in A_s from A for any $a \in \mathcal{A}(x)$, and so no number of X_s, the set X as measured immediately before the extraction of z from C, can leave X), implying that $z \in \Delta(X)$. Conversely, if $z \in \Delta(X)$ then $\Gamma(Y)[s] \subseteq X$, and since Step (5) never applies, we must have that $z \in C$ as desired.

Finally, note the situation for a $\tilde{\mathcal{T}}^{\tilde{c},\tilde{d}}$-strategy below the above $\mathcal{T}$-strategy: If the $\tilde{\mathcal{T}}$-strategy assumes a finite outcome (A) or (B) of the $\mathcal{T}$-strategy, then the $\tilde{\mathcal{T}}$-strategy acts exactly as the above $\mathcal{T}$-strategy; on the other hand, if the $\tilde{\mathcal{T}}$-strategy assumes the infinite outcome (C) of the $\mathcal{T}$-strategy, then this $\tilde{\mathcal{T}}$-strategy assumes that $\Gamma(Y)$ is finite; in fact, the $\tilde{\mathcal{T}}$-strategy will only be eligible to act at stages s at which the $\mathcal{T}$-strategy proceeds through Step (6) while $k \notin B_0$. The $\tilde{\mathcal{T}}$-strategy can now act as if in isolation, the restriction being that if $\tilde{c} \leq c$ then the $\tilde{\mathcal{T}}$-strategy can only use a witness with $\Theta_{\tilde{c},c}$-use in the stream S so as to keep Δ correct. (When the $\tilde{\mathcal{T}}$-strategy puts a number into, or extracts a number from, its set $\tilde{C}$ at a stage $\tilde{s}$, we also have to remove all other numbers $> \tilde{s}$ from the stream S of strategies below the $\tilde{\mathcal{T}}$-strategy and *dump* them into C since their assumption about the $\Phi(A)$'s may no longer be correct.)

Strategy for $\mathcal{T}^{c,d}$ below several $\mathcal{S}$-strategies: Again we may assume that $c \leq \mathcal{A}(x)$ and $d \geq \mathcal{B}(y)$; otherwise, the $\mathcal{T}$-strategy can diagonalize finitarily as in the previous case below a single $\mathcal{S}$-strategy.

But then the $\mathcal{T}$-strategy is essentially a nested version of the previous $\mathcal{T}$-strategy: If we only generate finitely many witnesses, or if we find a witness which is Γ-cleared for all Γ's above, then we will use it to diagonalize finitarily. Otherwise, we find the lowest-priority $\mathcal{S}$-requirement such that infinitely many witnesses are not Γ-cleared for its Γ. These witnesses will then constitute the stream S within which the strategies below this infinite outcome will have to work.

The possible *outcomes* of the above $\mathcal{T}$-strategy, in addition to the usual finitary outcomes, are then i_0 many infinitary outcomes (where i_0 is the number of Γ's that our $\mathcal{T}$-strategy has to deal with). We skip further details until the formal description of our construction.

Tree of strategies: We fix

$$\Lambda = \{\mathit{stop} <_\Lambda \infty_0 <_\Lambda \infty_1 <_\Lambda \cdots <_\Lambda \mathit{wait}\} \cup \{\infty <_\Lambda \mathit{fin}\}$$

as our set of outcomes, where ∞_i (for $i \in \omega$) as well as *stop* and *wait* are the possible outcomes of the $\mathcal{T}$-strategies; ∞ and *fin* are the possible outcomes of the $\mathcal{S}$-strategies; and $<_\Lambda$ denotes the ordering of the outcomes. (The ordering between $\{\infty, \mathit{fin}\}$ and the other outcomes is irrelevant since they will never be compared on the tree.) We use a tree $T \subseteq \Lambda^{<\omega}$ and refer to it as our *tree of strategies*. Each node of T will be associated with, and thus identified with, a *strategy*. We fix an effective *priority ordering* $\{\mathcal{R}_e\}_{e\in\omega}$ of all $\mathcal{S}$- and $\mathcal{T}$-requirements. (Requirement $\mathcal{P}$ and the $\mathcal{Q}$-requirements will be handled globally by the construction and thus do not appear on the tree of strategies.)

We now assign requirements to nodes on T by induction as follows: The empty node is *assigned* to requirement $\mathcal{R}_0$, and no requirement is *active* or *satisfied along* the empty node. Given an assignment to a node $\alpha \in T$, we distinguish cases depending on the requirement $\mathcal{R}$ assigned to α:

Case 1: $\mathcal{R}$ is an $\mathcal{S}$-requirement: Then call $\mathcal{R}$ *active along* $\alpha\hat{\ }\langle\infty\rangle$ *via* α, and call $\mathcal{R}$ *satisfied along* $\alpha\hat{\ }\langle \mathit{fin}\rangle$ *via* α. For all other requirements $\mathcal{R}'$ and for $o \in \{\infty, \mathit{fin}\}$, call $\mathcal{R}'$ *active* or *satisfied along* $\alpha\hat{\ }\langle o\rangle$ *via* $\beta \subset \alpha$ iff it is so along α, respectively. Now, for $o \in \{\infty, \mathit{fin}\}$, *assign* to $\alpha\hat{\ }\langle o\rangle$ the highest-priority requirement neither active nor satisfied along $\alpha\hat{\ }\langle o\rangle$. (The intuition is that under outcome *fin*, $\mathcal{R}$ is satisfied vacuously, whereas under outcome ∞, α builds an enumeration operator Γ.)

Case 2: $\mathcal{R}$ is a $\mathcal{T}^{c,d}$-requirement where $c \not\leq \mathcal{A}(x)$ or $d \not\geq \mathcal{B}(y)$: Then, for $o \in \{\mathit{stop}, \mathit{wait}\}$, call $\mathcal{R}$ *satisfied along* $\alpha\hat{\ }\langle o\rangle$ *via* α; and for all other requirements $\mathcal{R}'$, call $\mathcal{R}'$ *active* or *satisfied along* $\alpha\hat{\ }\langle o\rangle$ *via* $\beta \subset \alpha$ iff it is so along α, respectively. Now *assign* to $\alpha\hat{\ }\langle o\rangle$ (for $o \in \{\mathit{stop}, \mathit{wait}\}$) the highest-priority requirement neither active nor satisfied along $\alpha\hat{\ }\langle o\rangle$.

Case 3: $\mathcal{R}$ is a $\mathcal{T}^{c,d}$-requirement where $c \leq \mathcal{A}(x)$ and $d \geq \mathcal{B}(y)$: Let $\beta_0 \subset \dots \subset \beta_{i_0-1}$ be all the strategies $\beta \subset \alpha$ such that some $\mathcal{S}$-requirement is active along α via β_i. We denote by $\mathcal{S}_i$ the $\mathcal{S}$-requirement for β_i. (Here we allow $i_0 = 0$, in which case Case 3 will reduce to Case 2.) Then, for $o \in \{\mathit{stop}, \mathit{wait}\}$, call $\mathcal{R}$ *satisfied along* $\alpha\hat{\ }\langle o\rangle$ *via* α, and call any other requirement *active* or *satisfied along* $\alpha\hat{\ }\langle o\rangle$ *via* $\beta \subset \alpha$ iff it is so along α, respectively. Now (if $i_0 > 0$), fix $i \in [0, i_0)$. Call $\mathcal{S}_i$ *satisfied along* $\alpha\hat{\ }\langle\infty_i\rangle$ *via* α, and call any $\mathcal{S}_j$-requirement (for $j \in (i, i_0)$) neither *active* nor *satisfied along* $\alpha\hat{\ }\langle\infty_i\rangle$; call any other requirement *active* or *satisfied along* $\alpha\hat{\ }\langle\infty_i\rangle$ *via* $\beta \subset \alpha$ iff it is so along α, respectively. For any outcome $o \in \{\mathit{stop}, \mathit{wait}\} \cup \{\infty_i \mid i \in [0, i_0)\}$, *assign* to $\alpha\hat{\ }\langle o\rangle$ the highest-priority requirement neither active nor satisfied along $\alpha\hat{\ }\langle o\rangle$. (The intuition is that under the finitary outcomes *stop* and *wait*, the $\mathcal{T}$-requirement is assumed to be satisfied finitarily by diagonalization, whereas under outcome ∞_i, the $\mathcal{S}_i$-requirement, while previously satisfied via an enumeration operator Γ_i, is now assumed to be satisfied by α constructing an enumeration operator Δ_i, whereas all $\mathcal{S}_j$-requirements active via some strategy between β_i and α are assumed to be injured.)

The *tree of strategies* T is now the set of all nodes $\alpha \in \Lambda^{<\omega}$ to which requirements have been assigned.

As usual, our construction will let strategies be *eligible to act* if they are along the current approximation $f_s \in T$ to the true path $f \in [T]$ of the construction. So we call a stage s an *α-stage* if $\alpha \subseteq f_s$.

Construction: We proceed in stages $s \in \omega$.

We begin with some general conventions and definitions:

A strategy α is *initialized* by making all its parameters undefined and making the stream $S(\alpha)$ of α empty.

The stream $S(\emptyset)$ of the root node $\emptyset$ of our tree of strategies at any stage s is $[0, s)$ minus the set of numbers already dumped into C. The streams $S(\alpha)$ for $\alpha \neq \emptyset$ are defined during the construction.

At an α-stage s, call a number in the stream $S(\alpha)$ *suitable for* α if for every set C

(a) z is not currently *in use for* C by any strategy (i.e., z is not the current witness targeted for C of any $\mathcal{T}$-strategy, z is not a killing point or coding since z has been picked, and z is not the $\Theta_{c',c}$-use (for some $c' < c$) of any number);

(b) z has not been dumped into C;

(c) z is greater than $|\alpha|$ and greater than any stage at which any $\beta \supseteq \alpha$ has changed any set, picked any number, or extended any enumeration operator;

(d) z is greater than the stage s_β since which any $\beta \subset \alpha$ with finitary outcome $o \in \{\mathit{fin}, \mathit{stop}, \mathit{wait}\}$ along α has only taken outcome o;

(e) z is greater than z' many numbers in $S(\alpha)$ which are not in use for C by any $\beta \subseteq \alpha$ where z' is the greater of the last number in use by α and the most recent stage at which α was initialized;

(f) for any $e \in \mathcal{P} \cup \{y\}$ with $e > c$ and for any $\Theta_{c,e}$-axiom $\langle z, \{z'\}\rangle$, $z' \notin E$;

(g) for any $e \in \mathcal{P} \cup \{y\}$ with $e > c$, there is a number $z' \in S(\alpha)$ suitable for α (which thus can be chosen as a $\Theta_{c,e}$-use of z); and

(h) for any $d \in \mathcal{P}$ with $d < c$ and for any $\Theta_{d,c}$-axiom $\langle z', \{z\}\rangle$, z' has been dumped into D.

Note that the above definition of suitability is not circular since it proceeds by induction on the number of elements $e > c$. If the instructions below call for a strategy α to pick a number suitable for α and no such number currently exists, then we agree to simply end the stage. (We will show later that the collection of suitable numbers in any stream of a strategy along the true path is infinite, so the above delay will never be permanent. The purpose of the delay is to enforce automatic restraint, and to allow strategies below to have sufficiently many unused numbers available.)

Θ-*maintenance:* For the sake of the $\mathcal{Q}$-requirements, we guarantee during the construction (without explicitly mentioning this further) the following:

(1) When a number z is picked by a strategy α as a diagonalization witness, killing point, or coding number targeted for a set C (for $c \in \mathcal{P}$), then $z = z_c$ is suitable for α, and there is a (least) number z_e (for each $e \in \mathcal{P} \cup \{y\}$ with $e > c$) suitable for α. We then enumerate z_e into E (for each $e \geq c$) and add axioms $\langle z_{e_1}, \{z_{e_2}\}\rangle$ into Θ_{e_1,e_2} for all $e_1 \in \mathcal{P}$ and $e_2 \in \mathcal{P} \cup \{y\}$ with $c \leq e_1 < e_2$.

(2) When a number z (which was previously chosen, and is still in use now, by a strategy α) is enumerated into, or extracted from, a set C then we also enumerate z_e into, or extract z_e from, E, respectively (for all $e \in \mathcal{P} \cup \{y\}$ with $e > c$), where z_e is chosen as at the time z was picked by α.

(3) When a number z is a diagonalization witness, killing point, or coding number (targeted for a set C) of a strategy, that number is canceled, and that number has not been added to the stream of a strategy that is not initialized, then we *dump z into C* and add an axiom $\langle z, \emptyset\rangle$ into $\Theta_{c,e}$ for any $e \in \mathcal{P} \cup \{y\}$ with $e > c$. We also *dump into D* all z' which are targeted for D and for which a $\Theta_{c,d}$-axiom $\langle z, \{z'\}\rangle$ exists.

(4) When z is added to the stream $S(\alpha\hat{\ }\langle\infty_i\rangle)$ of a strategy $\alpha\hat{\ }\langle\infty_i\rangle$ (for some $\mathcal{T}^{c,d}$-strategy α) then z is extracted from C, and for any $e > c$ and for any $\Theta_{c,e}$-axiom $\langle z, \{z'\}\rangle$, we extract z' from E and remove z' from the stream $S(\alpha\hat{\ }\langle\infty_i\rangle)$ (so that the $\Theta_{c,e}$-axiom $\langle z, \{z'\}\rangle$ does not interfere with any future use of z).

All parameters are assumed to remain unchanged unless the construction below specifies otherwise.

Stage 0: We initialize all strategies.

Stage $s > 0$: Each stage s has substages $t \leq s$ such that some strategy $\alpha \in T$ of length t (with a "guess" about the outcomes of all strategies $\beta \subset \alpha$ that currently "look correct") acts at substage t of stage s and then decides which strategy will act at the next substage (or whether to end the stage). The longest strategy eligible to act at stage s is called the *current approximation to the true path* and is denoted by f_s. At the end of stage s (i.e., after all the substages of s), we initialize all strategies $\beta >_L f_s$.

Substage t of stage s: Suppose a strategy α of length t is eligible to act at this substage. We distinguish cases, depending on the requirement $\mathcal{R}$ assigned to α:

Case 1: $\mathcal{R}$ is an $\mathcal{S}_{X,\vec{\Phi}}$-requirement: Call (substage t of) stage s *α-expansionary* if α is initialized at s or if

$$\bigcup_{a \in \mathcal{A}(x)} \Phi_a(A)[s_0] \subseteq \bigcap_{a \in \mathcal{A}(x)} \Phi_a(A)[s],$$

where s_0 denotes (the end of) the most recent α-expansionary stage $< s$.

If s is not α-expansionary then end the substage by letting $\alpha\hat{\ }\langle fin \rangle$ be eligible to act next and setting the stream

$$S(\alpha\hat{\ }\langle fin \rangle) = S(\alpha) \cap [s_0, s).$$

Otherwise, for each z currently in $\bigcap_{a \in \mathcal{A}(x)} \Phi_a(A)$, let the *entry stage* e_z of z be the least stage such that for all $s' \in [e_z, s]$, $z \in \bigcap_{a \in \mathcal{A}(x)} \Phi_a(A)[s']$. Then choose z currently in $\bigcap_{a \in \mathcal{A}(x)} \Phi_a(A) - \Gamma(Y)$ (if any) with least entry stage (and the least z among these) and for each $b \in \mathcal{B}(y) - \{0\}$, pick a coding number z_b in $S(\alpha)$ suitable for α; enumerate z_b into B; and add an axiom $\langle z, F \rangle$ into Γ where the oracle set F is the set containing the $\Theta_{b,y}$-uses of all z_b's and $\Theta_{b_0,y}$-uses of all those current killing points k (with $k < e_z$) for Γ defined by $\mathcal{T}$-strategies $\beta \supseteq \alpha\hat{\ }\langle \infty \rangle$. Furthermore, for any z currently in $\Gamma(Y) - \bigcup_{a \in \mathcal{A}(x)} \Phi_a(A)$, the $\mathcal{S}$-strategy removes z_{b_0} from B_0. Now end the substage by letting $\alpha\hat{\ }\langle \infty \rangle$ be eligible to act next and setting the stream

$$S(\alpha\hat{\ }\langle \infty \rangle) = S(\alpha) \cap [s_0, s),$$

where s_0 is the most recent stage $\leq s$ at which α was initialized. (Recall that if a suitable number cannot be found, we end the stage.)

Case 2: $\mathcal{R}$ is a $\mathcal{T}^{c,d}_{\Psi}$-requirement: Let $\beta_0 \subset \cdots \subset \beta_{i_0-1}$ be all the strategies $\beta_i \subset \alpha$ such that some $\mathcal{S}_i$-requirement is active along α via β_i (allowing $i_0 = 0$). In the below, for $i \in [0, i_0)$, the enumeration operators $\Phi_{a,i}$ and Γ_i are those of β_i.

Now pick the first subcase which applies:

Case 2.1: α has not been eligible to act since its most recent initialization, or some number z had entry stage $e_z < k_i$ for X at the previous α-stage but now has entry stage $e_z > k_i$: We distinguish the following subcases:

Case 2.1.1: $c \leq \mathcal{A}(x)$; $d \geq \mathcal{B}(y)$; $i_0 > 0$; and the killing points k_i for $i \in [0, i_0)$ are currently undefined: Then pick killing points k_i suitable for α for each $i \in [0, i_0)$, and end the stage.

Case 2.1.2: $c \leq \mathcal{A}(x)$; $d \geq \mathcal{B}(y)$; $i_0 > 0$; and there is some number z which had entry stage $e_z < k_i$ for X at the previous α-stage but now has entry stage $e_z > k_i$: Then initialize all strategies $> \alpha$ and end the stage.

Case 2.1.3: Otherwise: End the stage.

Case 2.2: α currently has no witness z_{i_0}: Then pick a witness z_{i_0} suitable for α, add z_{i_0} to C, initialize all strategies $\beta \supseteq \alpha\hat{\ }\langle wait\rangle$, and end the stage.

Case 2.3: $z_{i_0} \in C - \Psi(D)$: Then end the substage by letting $\alpha\hat{\ }\langle wait\rangle$ be eligible to act next and setting the stream

$$S(\alpha\hat{\ }\langle wait\rangle) = S(\alpha) \cap [s_0, s)$$

where s_0 is the stage at which z_{i_0} was chosen.

Case 2.4: α has stopped (as defined below) and was not initialized since then: Then end the substage by letting $\alpha\hat{\ }\langle stop\rangle$ be eligible to act next and setting the stream

$$S(\alpha\hat{\ }\langle stop\rangle) = S(\alpha) \cap [s_0, s)$$

where s_0 is the stage at which α stopped.

Case 2.5: Otherwise: Then $z_{i_0} \in C \cap \Psi(D)$, so we call z_{i_0} a *realized* witness. We now need to distinguish further subcases, depending on the location of c and d relative to $\mathcal{A}(x)$ and $\mathcal{B}(y)$, respectively, and whether there are any active Γ's to worry about; so pick the first subcase which applies:

Case 2.5.1: $c \not\leq \mathcal{A}(x)$ or $i_0 = 0$: Then α *stops* by extracting z_{i_0} from C and ending the stage.

Case 2.5.2: $d \not\geq \mathcal{B}(y)$: Then α *stops* as follows: Fix some $b \in \mathcal{B}(y)$ with $b \not\leq d$. Now α extracts z_{i_0} from C, extracts the Γ_i-coding number z'_b from B for all $z' \in \Gamma_i(Y) - \bigcup_{a\in\mathcal{A}(x)} \Phi_{a,i}(A)$ and for each $i \in [0, i_0)$, repeating this process until $\Gamma_i(Y) \subseteq \bigcup_{a\in\mathcal{A}(x)} \Phi_{a,i}(A)$ for each $i < i_0$, and ends the stage.

Case 2.5.3: Otherwise: Then $c \leq \mathcal{A}(x)$; $d \geq \mathcal{B}(y)$; and $i_0 > 0$. For $i \in [0, i_0)$, call z Γ_i-*cleared* if

$$\Gamma_i(Y)[s_z] \subseteq \bigcup_{a\in\mathcal{A}(x)} \Phi_{a,i}(A - F_a),$$

where s_z is the stage at which z became a realized witness of α and F_a is the set of $\Theta_{c,a}$-uses of z.

Now α first extracts z_{i_0} from C. We then need to distinguish even further subcases, depending on whether we have a witness which is "fully Γ-cleared":

Case 2.5.3.1: Some witness z (current or former uncanceled, picked since α's most recent initialization) is Γ_i-cleared for all $i \in [0, i_0)$: Then α *stops* by removing z from C (if necessary), adding $B[s_z]$ into B for all $b \leq d$, setting $z_{i_0} = z$ as its current witness and ending the stage.

Case 2.5.3.2: Otherwise: We will now define the streams associated with α's infinitary outcomes. We will use z_i to denote the least element of the stream $S(\alpha\hat{\ }\langle\infty_i\rangle)$.

α acts as follows: Fix the least $i < i_0$ for which there is a current or former uncanceled witness z (minimal for this i, picked since α's most recent initialization) such that

$$z \notin S(\alpha\hat{\ }\langle\infty_i\rangle),$$

$$z \text{ is } \Gamma_j\text{-cleared for all } j \in (i, i_0), \text{ and}$$

$$z > \max\{z_j \mid j < i \text{ and } z_j \text{ currently defined}\}$$

(Here we set $\max(\emptyset) = 0$. Note that the above condition holds trivially for $z = z_{i_0}$ and $i = i_0 - 1$, so z defined as above must exist.)

Then α

(i) extract k_j (for each $j \in [i, i_0)$) from B_0 for the remainder of stage s;
(ii) cancels Δ_j for all $j \in (i, i_0)$;
(iii) cancels all (former or current) witnesses $z' \neq z$ of α with $z' \notin S(\alpha\hat{\ }\langle\infty_j\rangle)$ (for any $j \leq i$), makes z_j undefined for all $j \in (i, i_0]$ and makes α's current witness undefined;
(iv) sets $z_i = z$ if z_i is currently undefined;
(v) adds the axiom

$$\langle z, \Gamma_i(Y)[s_z]\rangle$$

into Δ_i;
(vi) adds axioms $\langle z', \emptyset\rangle$ into Δ_i for all z' with $z_i < z' < \max(S(\alpha\hat{\ }\langle\infty_i\rangle))$ and $z' \in C - S(\alpha\hat{\ }\langle\infty_i\rangle)$;
(vii) adds z to the stream $S(\alpha\hat{\ }\langle\infty_i\rangle)$, and
(viii) ends the substage by letting $\alpha\hat{\ }\langle\infty_i\rangle$ be eligible to act next.

Recall that by (4) of Θ-maintenance, z and all its Θ-uses will be removed from their respective sets.

Verification: Let $f = \liminf_s f_s$ be the *true path* of the construction, defined more precisely by induction as follows:

$$f(n) = \liminf_{\{s \mid f\restriction n \subset f_s\}} f_s(n).$$

Lemma 4.1 (Tree Lemma)**.**

(i) *Each $\alpha \subset f$ is initialized at most finitely often.*
(ii) *For each strategy $\alpha \subset f$, the stream $S(\alpha)$ is an infinite set. No number can leave $S(\alpha)$ unless α is initialized. For every $c \in \mathcal{P}$ and every stage s, there are an α-stage $t > s$ and a number $z > s$ such that z is suitable for α at stage t.*
(iii) *The true path f is an* infinite *path through T.*
(iv) *For any requirement $\mathcal{R}_e = \mathcal{S}_{X,\vec{\Phi}}$ or $\mathcal{T}_\Psi^{c,d}$, there is a strategy $\alpha \subset f$ such that the requirement is active via α along all sufficiently long $\beta \subset f$, or is satisfied via α along all β with $\alpha \subset \beta \subset f$. (In particular, for any requirement $\mathcal{R}_e$, there is a longest strategy assigned to $\mathcal{R}_e$ along f.)*

Proof. (i) Proceed by induction on α and note that there are only finitely many numbers with entry stage $\leq$ any fixed killing point k, so initialization via Case 2.1.2 of the construction is finite as long as the killing point k stabilizes.

(ii) Proceed by induction on $|\alpha|$ and note for the last part of (ii) that any number just entering $S(\alpha)$ is suitable for α at that stage.

(iii) A stage s is ended before substage s only under Cases 2.1, 2.5.1, 2.5.2, or 2.5.3.1 (which can apply at most twice after α's last initialization) and when a suitable number cannot be found (which cannot happen cofinitely often by (ii)).

(iv) By an easy induction argument on e. □

We now verify the satisfaction of the requirements.

Lemma 4.2 ($\mathcal{P}$ Lemma)**.** *Requirement $\mathcal{P}$ is satisfied.*

Proof. Any number z targeted for Y is the $\Theta_{b,y}$-use of a number z', say, targeted for B (for some $b \in \mathcal{B}(y)$). By the construction, we ensure $z \in Y$ iff $z' \in B$ from now on. So $\mathcal{P}$ is satisfied. $\square$

Lemma 4.3 ($\mathcal{Q}$-Lemma). *All requirements $\mathcal{Q}^{c,d}$ (for $c \in \mathcal{P}$ and $d \in \mathcal{P} \cup \{y\}$ with $c < d$) are satisfied. Also, 0 and 1 are mapped to $\mathbf{0}_e$ and $\mathbf{0}'_e$, respectively.*

Proof. By the definition of the images of $0, 1 \in \mathcal{P}$ and the remarks preceding the description of the requirements, the second part of Lemma 4.3 is clear.

So fix $c \in \mathcal{P}$ and $d \in \mathcal{P} \cup \{y\}$ with $0 < c < d < 1$ and any number z. By Θ-maintenance, whenever z is targeted for C (as diagonalization witness, killing point, coding number, or Θ-use) at a stage s, say, we also pick a $\Theta_{c,d}$-use z' of z. There are now three possibilities:

(i) α is initialized or z is canceled, in which case both z and z' are dumped;
(ii) z remains in the stream $S(\alpha)$ forever but never enters a stream $S(\alpha\hat{\ }\langle\infty_i\rangle)$, in which case $z \in C$ iff $z' \in D$; or
(iii) z enters a stream $S(\alpha\hat{\ }\langle\infty_i\rangle)$, at which time z is removed from C and z' is permanently removed from D (unless both are dumped later); in this case, if z later re-enters C, it will receive a new $\Theta_{c,d}$-use.

Finally note that (iii) above can apply only finitely often by clause (c) of the definition of suitability. $\square$

Lemma 4.4 ($\mathcal{T}$-Lemma). *All $\mathcal{T}$-requirements are satisfied.*

Proof. Fix a requirement $\mathcal{T}^{c,d}_{\Psi}$, and, by the Tree Lemma (Lemma 4.1(iv)), a $\mathcal{T}$-strategy $\alpha \subset f$ such that $\mathcal{T}^{c,d}_{\Psi}$ is satisfied via α along all sufficiently long $\beta \subset f$. Then $\alpha\hat{\ }\langle o\rangle \subset f$ for $o \in \{\mathit{stop}, \mathit{wait}\}$.

By the construction and the fact that α is eventually no longer initialized, α eventually has a fixed diagonalization witness z, say.

If $\alpha\hat{\ }\langle \mathit{wait}\rangle \subset f$ then $z \in C - \Psi(D)$ by the construction, thus the requirement $\mathcal{T}^{c,d}_{\Psi}$ is clearly satisfied.

Otherwise, $\alpha\hat{\ }\langle \mathit{stop}\rangle \subset f$, so α stops at a stage s, say; and $z \in \Psi(D)[s] - C$. We will show that *no* set changes at any number $< s_z$ (where s_z is the stage $\leq s$ at which z becomes a realized witness) by considering all possible strategies β:

Case A: $\beta <_L \alpha$: Then β is no longer eligible to act after stage s (or else α would be initialized and lose its witness).

Case B: $\beta \geq \alpha\hat{\ }\langle \mathit{stop}\rangle$: The first time β is eligible to act after α stops, β acts the first time after being initialized and thus cannot change D at a number that will injure $\Psi(D; z)$.

Case C: $\beta\hat{\ }\langle o\rangle \subseteq \alpha\hat{\ }\langle \mathit{stop}\rangle$ for $o \in \{\mathit{stop}, \mathit{wait}, \mathit{fin}\}$: Then β cannot change D without initializing α.

Case D: $\beta\hat{\ }\langle\infty_i\rangle \subseteq \alpha$ for some $i \in \omega$: Then z was put by β into the stream of $\beta\hat{\ }\langle\infty_i\rangle$, and at stage s, β adds a number $> z$ into the stream of $\beta\hat{\ }\langle\infty_i\rangle$. At the first β-stage $> s$, β picks a diagonalization witness z' as well as its Θ-uses, all of which are too large to injure $\Psi(D; z)$, and after stage s, β does not change D at a number less than z' and its Θ-uses. So β cannot injure $\Psi(D; z)$ after stage s.

Case E: $\beta\hat{\ }\langle\infty\rangle \subseteq \alpha$ and β's $\mathcal{S}$-requirement is active along α via β: Then α stops

2.5.3.2. Thus z is Γ_i-cleared, i.e.,

$$\Gamma_i(Y)[s_z] \subseteq \bigcup_{a \in \mathcal{A}(x)} \Phi_{a,i}(A - F_a),$$

where F_a is the set of $\Theta_{c,a}$-uses of z. By the action at stage s,

$$\Gamma_i(Y)[s] \subseteq \bigcup_{a \in \mathcal{A}(x)} \Phi_{a,i}(A)[s],$$

so any later Γ_i-correction by β will only involve Γ_i-axioms defined after stage s_z, and thus will change any set only on numbers $> s_z$.

Case F: $\beta\hat{\ }\langle\infty\rangle \subseteq \alpha$ *and* β*'s* $\mathcal{S}$*-requirement is not active along* α *via* β: Then some α' with $\beta \subset \alpha' \subset \alpha$ kills β's enumeration operator Γ. Then

$$\Gamma(Y)[s_z] \subseteq \bigcup_{a \in \mathcal{A}(x)} \Phi_a(A)[s_z]$$

by the action of β at stage s_z; and any later Γ-correction by β will only involve Γ-axioms defined after stage s_z, and thus will change any set only on numbers $> s_z$. □

Lemma 4.5 ($\mathcal{S}$-Lemma)**.** *All* $\mathcal{S}$*-requirements are satisfied.*

Proof. Fix a requirement $\mathcal{S}_{X,\vec{\Phi}}$. Assume that $\Phi_{a_0}(A_0) = \Phi_{a_1}(A_1)$ for all $a_0, a_1 \in \mathcal{A}(x)$, and denote their common value by X. Then by the Tree Lemma (Lemma 4.1(iv)), there is a longest $\mathcal{S}_{X,\vec{\Phi}}$-strategy $\beta \subset f$. We argue that there are infinitely many β-expansionary stages, so $\beta\hat{\ }\langle\infty\rangle \subset f$. Otherwise, let s_0 be the greatest β-expansionary stage. Then no Φ-computation existing at the end of stage s_0 can be destroyed after stage s_0 since any strategy $>_L \alpha\hat{\ }\langle\infty\rangle$ cannot remove a number $\leq s_0$, and neither can any strategy $\subset \beta$ as in Cases C through F of Lemma 4.4.

Again by the Tree Lemma (Lemma 4.1(iv)), we may now distinguish two cases:

Case 1: $\mathcal{S}_{X,\vec{\Phi}}$ *is active via* β *along all* α *with* $\beta \subset \alpha \subset f$: Suppose that β is no longer initialized after stage s_0, say.

For the sake of a contradiction, assume first that $z \in X - \Gamma(Y)$ for some z of least entry stage s_z. Fix a stage $s_1 \geq s_0, s_z$ such that all $z' \in X$ with lesser entry stage are permanently in $\Gamma(Y)$. Fix $s_2 \geq s_1$ such that no $\mathcal{T}$-strategy with killing point $\leq z$ (for this Γ) executes step (i) of Case 2.5.3.2 of the construction. Then by the first β-expansionary stage $\geq s_2$, β will permanently put z into $\Gamma(Y)$ by Case 1 of the construction.

On the other hand, suppose $z \in \Gamma(Y)$. Then by Γ-correction by β under Case 1 of our construction, $z \in X$ as desired.

Case 2: There is a $\mathcal{T}_{\Psi}^{c,d}$-strategy $\alpha \subset f$ such that $\mathcal{S}_{X,\vec{\Phi}}$ is satisfied via α along all ξ with $\beta \subset \xi \subset f$: Then β is α's strategy β_i, $\alpha\hat{\ }\langle\infty_i\rangle \subset f$, and we need to show that $\Delta_i(X) = C$ (for the enumeration operator Δ_i built by α after α's last initialization and after α cancels Δ_i for the last time).

We distinguish two cases for arguments z of $\Delta_i(X)$:

Case 2A: $z \notin S(\alpha\hat{\ }\langle\infty_i\rangle)$: Then, once $z < \max(S(\alpha\hat{\ }\langle\infty_i\rangle)[s])$, no strategy can remove z from C (and so by (v) of Case 2.5.3.2 of the construction, $z \in C$ iff $z \in \Delta_i(X)$). To see this, note that only strategies $\xi \subset \alpha$ with infinitary outcome along α can possibly change $C(z)$ (by the usual initialization argument). But, after stage s, any such ξ cannot put z into the stream of any strategy $\supset \xi$. If ξ is a $\mathcal{T}$-strategy, it will no longer remove z as a realized witness, and it will not remove z for Γ-correction (as in Case 2.5.2 of the construction) since ξ does not stop (as Case 2.5.2 does not apply). If ξ is an $\mathcal{S}$-strategy then ξ only removes numbers from B_0 but $b_0 \not\leq c$.

Case 2B: $z \in S(\alpha\hat{\ }\langle\infty_i\rangle)$: We first observe that

$$z \in C \Leftrightarrow F_a \subseteq A \text{ for all } a \in \mathcal{A}(x), \text{ and}$$

$$z \notin C \Leftrightarrow F_a \cap A = \emptyset \text{ for all } a \in \mathcal{A}(x), \tag{1}$$

$$z \in \Delta_i(X) \Leftrightarrow \Gamma_i(Y)[s_z] \subseteq X, \text{ and} \tag{2}$$

$$\Gamma_i(Y)[s_z] \not\subseteq \bigcup_{a \in \mathcal{A}(x)} \Phi_{a,i}(A - F_a) \tag{3}$$

by Θ-maintenance, the definition of Δ_i, and the fact that α does not stop, respectively.

Thus $z \notin C$, by (1), implies

$$\bigcup \Phi_{a,i}(A - F_a) = X,$$

so, by (3), we have

$$\Gamma_i(Y)[s_z] \not\subseteq X,$$

which, by (2), entails $z \notin \Delta_i(X)$ as desired.

On the other hand, if $z \in C$ then, by (1),

$$F_a \subseteq A \text{ for all } a \in \mathcal{A}(x),$$

so, since s_z is β_i-expansionary,

$$\Gamma_i(Y)[s_z] \subseteq \bigcup_{a \in \mathcal{A}(x)} \Phi_{a,i}(A \cup F_a)[s_z] \subseteq \bigcup_{a \in \mathcal{A}(x)} \Phi_{a,i}(A) = X,$$

implying $z \in \Delta_i(X)$ as desired. □

This completes the proof of the third non-extendibility theorem.

5. The extendibility theorem. In this section, we show that if conditions (1), (2), and (3) of Theorem 1.5 fail, then extendibility must hold.

We assume the following is known to the reader. Suppose that $(W_i : i \in \omega)$ is a sequence of Σ^0_2 subsets of ω, and that $(W_i[s] : i \in \omega \text{ and } s \in \omega)$ is a uniformly computable approximation to it. That is, for each $i \in \omega$ and for all n, $n \in W_i$ if and only if there are cofinitely many s such that $n \in W_i[s]$. Then $(W_i : i \in \omega)$ has a uniformly computable approximation $(W^*_i[s] : i \in \omega \text{ and } s \in \omega)$ such that for each s, $W^*_i[s]$ is a subset of the intersection of $W_i[s]$ with the numbers less than s, and such that there are infinitely many s such that for all n and all i, $n \in W^*_i[s]$ if and only if $n \in W_i$. Such an s is called a *true stage* in the approximation. Further, the computable description of the *-sequence may be obtained effectively from a computable description of the original one. In the following, we will be given finitely many Σ^0_2 sets and will give computable approximations for finitely many others.

By the Kleene recursion theorem, we may assume that we have an index for this sequence when we begin our construction. Thus, we may assume that we have a computable approximation to the sequence of sets given to us together with the ones that we enumerate such that there are infinitely many true stages for that approximation.

We will be given sets A, B, C, and we will construct sets X, Y, U, and so forth. By this, we mean that we will present computable approximations to the sets that we construct. For each set X under construction, $X[0]$ is empty. When at a stage s, we put a number n into X, we are setting $X(n)[s]$ to 1. Similarly, if we extract a number from X, we are setting $X(n)[s]$ to 0. To fix our notation, the **-approximation* of these sets is the one referred to in the previous paragraph, with infinitely many true stages.

We now proceed to the proof of the extendibility theorem.

Definition 5.1. Suppose that $\mathcal{P} \subseteq \mathcal{Q}$ are finite partially ordered sets. The *extension conditions* for $\mathcal{Q}$ over $\mathcal{P}$ are the following three conditions.

(1) If $x \not\leq y$ in $\mathcal{Q}$, then $\mathcal{BA}(x) \not\subseteq \mathcal{BAB}(y)$.
(2) If $y \in \mathcal{Q} - \mathcal{P}$, then $\mathcal{Z}(y) = \emptyset$ or $\mathcal{BA}(\mathcal{Z}(y) \cup \mathcal{B}(y)) \subseteq \mathcal{B}(y)$, where $\mathcal{Z}(y)$ is given by

$$\mathcal{Z}(y) = \{z : z \in \mathcal{Q} - \mathcal{P},\ z < y,\ \text{and } \mathcal{B}(y) \not\subseteq \mathcal{BA}(z)\}.$$

(3) If $x \not\leq y$ in $\mathcal{Q}$, then either $\mathcal{B}(x) \not\subseteq \mathcal{B}(y)$ or $\mathcal{B}(y) \subseteq \mathcal{BA}(x)$.

(Notice that Condition 3 is obviously true when $x \in \mathcal{P}$.)

Theorem 5.2. *Suppose that $\mathcal{P} \subseteq \mathcal{Q}$ are finite bounded partial orders such that $\mathcal{Q}$ satisfies all three of the extension conditions over $\mathcal{P}$. Then for any π, a partial order embedding of $\mathcal{P}$ into the Σ^0_2-enumeration degrees preserving 0 and 1, there is an extension of π to a partial order embedding of $\mathcal{Q}$, also preserving 0 and 1.*

Definition 5.3. Suppose that $\mathcal{P}$ is a finite bounded partial order and that $\mathcal{Q}$ is a finite extension of $\mathcal{P}$. We say that $\mathcal{J}$ is a *minimal extension ideal* if there is a $u \in \mathcal{Q} - \mathcal{P}$ such that $\mathcal{J} = \mathcal{BA}(u)$ and there is no proper subset of $\mathcal{J}$ with the same property.

Suppose that $\mathcal{P}$ is a finite bounded partial order, $\mathcal{Q}$ is a finite extension of $\mathcal{P}$ satisfying the extension conditions, π is an embedding of $\mathcal{P}$ into the Σ^0_2-enumeration degrees, and $\mathcal{J}$ is a minimal extension ideal in $\mathcal{P}$. Let $R^{\mathcal{J}}$ be the subset of $\mathcal{Q} - \mathcal{P}$ given by $R^{\mathcal{J}} = \{u : u \in \mathcal{Q} - \mathcal{P} \text{ and } \mathcal{BA}(u) = \mathcal{J}\}$. We let $\mathcal{Q}^{\mathcal{J}} = \mathcal{P} \cup R^{\mathcal{J}}$ with the order obtained by restriction from $\mathcal{Q}$.

Continuing, suppose that u_1 and u_2 belong to $\mathcal{Q} - \mathcal{P}$ and $u_1 > u_2$. Then, $\mathcal{BA}(u_2) \subseteq \mathcal{BA}(u_1)$. Consequently, if $u_1 \in R^{\mathcal{J}}$, then by the minimality of $\mathcal{J}$, $\mathcal{BA}(u_2) = \mathcal{J}$. Thus the elements of $R^{\mathcal{J}}$ are closed downward in $\mathcal{Q} - \mathcal{P}$. Further if u_1 and u_2 are elements of $R^{\mathcal{J}}$, then because $\mathcal{BA}(u_1) = \mathcal{BA}(u_2) = \mathcal{J}$, $u_2 \notin \mathcal{Z}(u_1)$.

In the following, we will use lower case letters (d, x, y, u an so forth) to denote elements of the finite partially ordered set $\mathcal{Q}$ and the same letters in upper case to denote the sets corresponding to the images (namely, Σ^0_2-enumeration degrees) of these points. We sometimes identify the set corresponding to the image of a point, with the image itself, thus writing for instance $X = \pi(x)$. Analogously, $\mathcal{B}(Y)$ will be the set of images of elements of $\mathcal{B}(y)$. Likewise, $\mathcal{J}$ will sometimes denote the sets corresponding to the images of elements in the minimal extension ideal $\mathcal{J}$: the exact meaning of the symbol will be clear from the context.

We will extend π to an embedding of $\mathcal{Q}$ by a sequence of extensions. During each step of the sequence, we will extend the domain of our embedding to include the elements of a minimal extension ideal over the previous embedding. We will ensure that $\mathcal{Q}$ satisfy a version of the extension conditions.

Definition 5.4. Suppose that $\mathcal{P} \subseteq \mathcal{Q}$ are finite partially ordered sets with 0 and 1 and π is an embedding of $\mathcal{P}$ into the Σ^0_2-enumeration degrees. The *extension conditions for $\mathcal{Q}$ over $\mathcal{P}$ and π* are the following three conditions.

(1) If $x \not\leq y$ in $\mathcal{Q}$, then there is an h in $\mathcal{BA}(x)$ such that $H \not\leq_e \bigoplus \mathcal{B}(Y)$.
(2) If $y \in \mathcal{Q} - \mathcal{P}$, then $\mathcal{Z}(y) = \emptyset$ or $\mathcal{BA}(\mathcal{Z}(y) \cup \mathcal{B}(y)) \subseteq \mathcal{B}(y)$, where again $\mathcal{Z}(y)$ is given by

$$\mathcal{Z}(y) = \{z : z \in \mathcal{Q} - \mathcal{P},\ z < y, \text{ and } \mathcal{B}(y) \not\subseteq \mathcal{BA}(z)\}.$$

(3) If $x \not\leq y$ in $\mathcal{Q}$, then either $\mathcal{B}(x) \not\subseteq \mathcal{B}(y)$ or $\mathcal{B}(y) \subseteq \mathcal{BA}(x)$.

Note that if $\mathcal{P} \subseteq \mathcal{Q}$ satisfy the extension conditions of Definition 5.1 and π is an embedding of $\mathcal{P}$ into the Σ^0_2-enumeration degrees then $\mathcal{Q}$ satisfies the extension conditions over $\mathcal{P}$ and π as in Definition 5.4.

In the following let us use the symbol $\mathcal{E}(\Sigma^0_2)$ to denote the poset of Σ^0_2-enumeration degrees.

Proposition 5.5. *For any π, $\mathcal{P}$, $\mathcal{Q}$ such that $\mathcal{Q}$ satisfies the extension conditions over $\mathcal{P}$ and π, and any $\mathcal{J}$ a minimal extension ideal in $\mathcal{Q}$ over $\mathcal{P}$, there is an extension $\pi^{\mathcal{J}} : \mathcal{Q}^{\mathcal{J}} \to \mathcal{E}(\Sigma^0_2)$ of π such that the following conditions hold.*

(1) *For all u in $R^{\mathcal{J}}$, $\bigoplus \mathcal{J} \geq_e U$.*
(2) *For all $y \in \mathcal{Q}$ and $d \in \mathcal{P}$, if $\vec{u}$ is the set of elements u in $R^{\mathcal{J}}$ such that $y \geq u$, and if $y \not\geq d$, then $\bigoplus \vec{U} \oplus \bigoplus \mathcal{B}(Y) \not\geq_e D$.*
(3) *For all $y \in \mathcal{Q}$ and $v \in R^{\mathcal{J}}$, if $\vec{u}$ is the set of elements u in $R^{\mathcal{J}}$ such that $y \geq u$, and if $y \not\geq v$, then $\bigoplus \vec{U} \oplus \bigoplus \mathcal{B}(Y) \not\geq_e V$.*

Note that second and third conditions ensure that if y and x are elements of $\mathcal{Q}^{\mathcal{J}}$ and $y \not\geq x$, then $Y \not\geq_e X$, where $Y = \pi^{\mathcal{J}}(y)$ and $X = \pi^{\mathcal{J}}(x)$.

5.1. Strategies. We fix enumerations of the finite sets $x_1, x_2, \ldots$ of elements of $\mathcal{P}$ and $u_1, u_2, \ldots$ of elements of $R^{\mathcal{J}}$. We let X_i denote the set corresponding to the image $\pi(x_i)$ of x_i and let U_i denote the Σ^0_2 set that we will construct as the image of u_i.

5.2. Comparability strategies. We fix a uniformly computable sequence $(R_i : i \in \omega)$ of infinite, pairwise disjoint computable subsets of ω.

We choose the embedding π of $\mathcal{P}$ into $\mathcal{E}(\Sigma^0_2)$ so that the ith element of $\mathcal{P}$ is mapped to a subset of R_i.

If $y > x$ in $\mathcal{Q}^{\mathcal{J}}$ and $y \in R^{\mathcal{J}}$ then we ensure that $Y \geq_e X$ by requiring that $X \subseteq Y$. Now, X will be contained in the (computable) union of those R_i which are used to code elements of $\pi(\mathcal{P})$ into X together with those R_i used by Sacks coding strategies directing numbers into X and sets below X. No other numbers from this union will belong to Y. And so, n will be an element of X if and only if n is an element of the above computable union and $n \in Y$.

If $y > x$ in $\mathcal{Q}^{\mathcal{J}}$ and $y \in \mathcal{P}$ then we distinguish two cases: If $x \in \mathcal{P}$ then $Y \geq_e X$ since π is an embedding. If $x \in R^{\mathcal{J}}$ then $y \geq \mathcal{BA}(x) = \mathcal{J}$, and so $Y \geq_e \bigoplus \mathcal{J}$, and by clause (1) of Proposition 5.5, $Y \geq_e X$.

5.3. Incomparability strategies. We have two families of incomparability requirements, corresponding to Clauses 2 and 3 of Proposition 5.5. We will present strategies to ensure that $\Theta(\bigoplus \vec{U} \oplus \bigoplus \mathcal{B}(Y)) \neq X$, treating these three cases separately: $X \in \pi(\mathcal{P})$, $\vec{U}$ is empty (a special case of Clause 3 in Proposition 5.5), and neither of the previous two. Of course, since we are given an embedding of $\mathcal{P}$, if $\vec{U}$ is empty and $X \in \pi(\mathcal{P})$, then since $y \not\geq x$ in $\mathcal{Q}$, $\bigoplus \mathcal{B}(Y) \not\geq_e X$ (apply Condition (1) in Definition 5.1).

Before we describe the strategies, we anticipate their success and describe the parameter which characterizes their outcomes.

Definition 5.6. Suppose that Y and X are Σ^0_2 sets such that for all $n \in \omega$,

$$\begin{aligned} n \in Y &\iff (\exists u)(\forall v)\Phi_Y(u, v, n) \\ n \in X &\iff (\exists u)(\forall v)\Phi_X(u, v, n) \end{aligned}$$

in which Φ_Y and Φ_X are computable predicates, and suppose that $\Xi(Y) \neq X$. Then, w is a *witness to the inequality* (between $\Xi(Y)$ and X) if either one of the following conditions (1) or (2) holds.

(1) $w = \langle 1, n, \langle x_1, s_1\rangle, \ldots, \langle x_m, s_m\rangle, t\rangle$ and

- $(\forall i \leq m)(\forall u < s_i)(\exists v < t)\neg\Phi_Y(u, v, x_i)$,
- $(\forall i \leq m)(\forall v)\Phi_Y(s_i, v, x_i)$, $\langle n, \{x_1, \ldots, x_m\}\rangle \in \Xi[t]$
- $n \notin X$.

So, $n \in \Xi(Y) - X$.

(2) $w = \langle 0, n, s, t\rangle$, and

- $(\forall u < s)(\exists v < t)\neg\Phi_X(u, v, n)$,
- $(\forall v)\Phi_X(s, v, n)$,
- $\Xi(Y, n)$ diverges.

So, $n \in X - \Xi(Y)$.

Note that in both cases, w's being a witness to $\Xi(Y) \neq X$ is expressed by the conjunction of a Π^0_1 formula with a Π^0_2 formula. We will abbreviate the conjunction as $(\forall x)\phi_1(x, w)$ & $(\forall x)(\exists y)\phi_2(x, y, w)$.

Lemma 5.7. *Suppose that Y and X are enumeration reducible to A. Then the condition "no number less than w is a witness to $\Xi(Y) \neq X$" is enumeration reducible to A, uniformly in Ξ.*

Proof. A number w is not a witness to $\Xi(Y) \neq X$ if either a Σ^0_1 condition holds, which is enumerable relative to A, or either a number belongs to a set X enumeration reducible to A (Clause (1) of Definition 5.6) or a computation $\Xi(n, Y)$ converges using only positive information about a set enumeration reducible to A (Clause (2) of Definition 5.6). Both of the latter disjuncts are enumeration reducible to A. □

Approximating the least witness to $\Xi(Y) \neq X$. Let $w(1), w(2), \ldots$ be an enumeration of the possible witnesses to $\Xi(Y) \neq X$, written in increasing order. Let ϕ_1 and ϕ_2 be taken as in the comment following Definition 5.6. Note, we are using the fixed point theorem to work with ϕ_1 and ϕ_2 during the construction of the sets to which they refer.

We define the following parameters for the strategy during stage s. For each k, define $\ell(k)[s]$ to be the largest ℓ such that $(\forall x \leq \ell)(\exists y \leq s)\phi_2(x, y, w(k))$. Let $w[s]$ be the least $w(k)$ such that

(1) $(\forall x \leq s)\phi_1(x, w(k))$,
(2) $(\forall t < s)(\ell(k)[t] < \ell(k)[s])$.

Programming environment. Ultimately, we will run our diagonalization strategies simultaneously. With this in mind, we will design our strategies to work within the constraints that we expect to encounter.

Constraints on σ_i. The constraint within which we will be working for the ith incomparability strategy σ_i will be of the following type. Suppose that σ_i is meant to satisfy the requirement $\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i)) \neq X_i$. We are using the following notation: S_i is the set of indices of sets U_j with $u_j \leq y_i$ in $\mathcal{Q}$, and $\bigoplus_{S_i} U_j$ is the join of those U_j's whose indices belong to S_i. In the course of running σ_i, we will approximate the least witness w_i to the inequality between a functional constructed by σ_i relative to a join of sets from $\mathcal{P}$ and another element of $\mathcal{P}$.

We let s denote a stage in the execution of σ_i.

(1) σ_i must be consistent with the coding requirements discussed earlier.
(2) σ_i cannot designate that any numbers from $\bigcup_{j<i} R_j$ belong to any set. That is, $\bigcup_{j<i} R_j$ will be controlled by the strategies of higher priority.
(3) Let $\vec{I_i}$ be the Σ_2^0 sequence of sets of length $|R^{\mathcal{J}}|$, the cardinality of $R^{\mathcal{J}}$, such that for every $m < |R^{\mathcal{J}}|$, the numbers which are eventually put into U_m by a strategy of higher priority than σ_i are exactly those numbers in $\vec{I_i}(m)$, the mth element of $\vec{I_i}$. We say that $\vec{I_i}(m)$ is the injury set for U_m.

    σ_i will be given approximations $(\vec{I_i}[s] : s \in \omega)$ to $\vec{I_i}$, $\vec{U}[s]$ to $\vec{U}$, $\mathcal{B}(Y_i)[s]$ to $\mathcal{B}(Y_i)$, and $X[s]$ to X.

    σ_i is constrained so that for all $n \in R_j$ and U under construction, if numbers from R_j are designated to belong to U and there is an m such that $n \in \vec{I_i}(m)[s]$, then $n \in U[s]$. The set to which numbers are designated will be clear from the construction.
(4) σ_i will be given simultaneous true stage approximations for all of $\vec{I_i^*}[s]$, $\vec{U}^*[s]$, $\mathcal{B}(Y_i)^*[s]$, $X_i^*[s]$, and $w_i^*[s]$. The first four correspond to the sets mentioned in the previous item. The last corresponds to our approximation to the least witness w_i as described earlier. We may assume that for all stages s, $\vec{I_i^*}[s] \subseteq \vec{I_i}[s]$, $\vec{U}^*[s] \subseteq \vec{U}[s]$, $\mathcal{B}(Y_i)^*[s] \subseteq \mathcal{B}(Y_i)[s]$, and $w_i^*[s] \leq w_i[s]$.

    There are infinitely many stages s such that $\vec{I_i^*}[s] \subseteq \vec{I_i}$, $\vec{U}^*[s] \subseteq \vec{U}$, $\mathcal{B}(Y_i)^*[s] \subseteq \mathcal{B}(Y_i)$, and $X_i^*[s] \subseteq X_i$. Further, if $\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i)) \neq X_i$, then for infinitely many of these stages s, $w_i^*[s]$ is the least witness to this inequality.

Of course, we will design our strategies so that their only permanent effect is to impose a system of constraints as above.

In our construction, we will have incomparability strategies for $y \not\geq x$ only when $\mathcal{B}(y) \subseteq \mathcal{J}$. The satisfaction of the other incomparability requirements will follow by an algebraic argument.

Case 1: Requirement $\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i)) \neq D_i$ with $\mathcal{B}(y_i) \subseteq \mathcal{J}$ and $d_i \in \mathcal{P}$. If d_i is not an element of $\mathcal{J}$, then we may conclude that $\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i) \not\geq_e$ D_i once we prove that $\bigoplus_{S_i} U_j$ is below $\bigoplus \mathcal{J}$. If $d_i \notin \mathcal{J}$, then we say that

$\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i)) \neq D_i$ is a trivial requirement.

When $d_i \in \mathcal{J}$, then we implement a version of the Sacks preservation strategy as follows.

We will enumerate a functional Γ_i and ensure

$$\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i)) = D_i \implies \Gamma_i(\bigoplus_{S_i} \vec{I}_i(j) \oplus \bigoplus \mathcal{B}(Y_i)) = D_i.$$

Sacks preservation. We order finite sets by canonical index, i.e., for finite sets G_0 and G_1, say that G_0 is less than G_1 if the greatest element of G_0 is less than the greatest element of G_1 or if their greatest elements are equal and G_0 precedes G_1 lexicographically. Say that a finite set G *appears consistent with* $\bigoplus_{S_i} \vec{I}_i(j) \oplus \bigoplus \mathcal{B}(Y_i)$ *during stage* s, if G is of the form $\bigoplus \vec{H} \oplus \bigoplus \vec{K}$, where $\vec{H}$ and $\vec{K}$ are sequences of finite sets of the same lengths as $\vec{I}_i$ and $\mathcal{B}(Y_i)$, for any H_m in $\vec{H}$, $H_m \cap \bigcup_{j<i} R_j$ is a subset of those numbers in $\vec{I}_i^*(j_m)[s]$, where j_m is the mth element of S_i and $\vec{I}_i^*(j_m)[s]$ is the mth component of $\vec{I}_i^*[s]$, and $\vec{K}$ is contained in $\mathcal{B}(Y_i)^*[s]$. In short, the numbers appearing in G which are under the control of the strategies of higher priority all appear to be in their designated sets. For example, we may assume that $\bigoplus_{S_i} U_j^* \oplus \bigoplus \mathcal{B}(Y_i)^*$ appears consistent with $\bigoplus_{S_i} \vec{I}_i(j) \oplus \bigoplus \mathcal{B}(Y_i)$ during every stage s.

Let w_i be the least witness to the inequality between $\Gamma_i(\bigoplus_{S_i} \vec{I}_i(j) \oplus \bigoplus \mathcal{B}(Y_i))$ and D_i, with the understanding that we will later prove $\Gamma_i(\bigoplus_{S_i} \vec{I}_i(j) \oplus \bigoplus \mathcal{B}(Y_i))$ not equal to D_i. The strategy σ_i works with the approximation $w_i[s]$, and does not depend on the value of w_i, or even on whether $\Gamma_i(\bigoplus_{S_i} \vec{I}_i(j) \oplus \bigoplus \mathcal{B}(Y_i))$ is actually unequal to D_i.

During stage s, we take the following action. If $s = 0$, then we have no constraints from earlier stages.

Description. Enumerating Γ_i. During each stage s, for each n such that $n \in D_i^*[s]$ and $\Theta_i(\bigoplus_{S_i} U_j^* \oplus \bigoplus \mathcal{B}(Y_i)^*; n)[s] \downarrow$, if $\Gamma_i(\bigoplus_{S_i} \vec{I}_i(j)^* \oplus \bigoplus \mathcal{B}(Y_i)^*; n)[s] \uparrow$, then let $\langle n, G_0 \oplus B_0 \rangle$ be the least axiom in $\Theta_i[s]$ (i.e., the axiom with least Gödel number) such that $G_0 \oplus B_0$ appears consistent with $\bigoplus_{S_i} \vec{I}_i(j)^* \oplus \bigoplus \mathcal{B}(Y_i)^*$ during stage s, and enumerate the axiom $\langle n, \bigoplus_{S_i} \vec{I}_i(j)^*[s] \oplus B_0 \rangle$ into Γ_i. We *associate* the axiom $\langle n, \bigoplus_{S_i} \vec{I}_i(j)^*[s] \oplus B_0 \rangle$ in Γ_i with $\langle n, G_0 \oplus B_0 \rangle$. (More formally, the set of associated pairs of axioms is a computably enumerable set generated simultaneously with Γ_i.)

Imposing constraints. For each n and for each axiom $\langle n, F \rangle$ in Γ_i, let $\langle n, G_0 \oplus B_0 \rangle$ be the axiom in Θ_i associated with $\langle n, F \rangle$. We impose the constraint that $G_0 \subseteq \bigoplus_{S_i} U_j$ *for the sake of* $\langle n, F \rangle$. By this we mean, for each number m in G_0 which is not in $\bigoplus_{S_i} \vec{I}_i(j)$, either for all sufficiently large stages t the number m belongs to $\bigoplus_{S_i} U_j[t]$, or there are infinitely many stages during which we postpone the constraint imposed for the sake of $\langle n, F \rangle$ (as defined below). Here, we arrange that $G_0 \subseteq \bigoplus_{S_i} U_j$ by adding numbers to the sets U_j, for $j \in S_i$. This constraint may involve the inclusion of these same numbers into other sets for the coding.

Postponing constraints. Postpone all constraints imposed for the sake of axioms $\langle n, F \rangle$ for which $n \geq w_i^*[s]$. Additionally, postpone any constraint imposed for the sake of an axiom $\langle n, F \rangle \in \Gamma_i$ such that F does not appear consistent with $\bigoplus_{S_i} \vec{I}_i(j) \oplus \bigoplus \mathcal{B}(Y_i)$.

We have used the word "postpone" without having defined it. Our intention is that a constraint that is postponed infinitely often is not meant to apply in the limit, and that a constraint that is only postponed finitely often is meant to apply for every stage after the last one in which it was postponed.

Deciding values of Y_i. Once the needed constraints have been enumerated and the ones that seem superfluous have been postponed, for every constraint "$G_0 \subseteq \bigoplus_{S_i} U_j$ for the sake of $\langle n, F\rangle$" that σ_i has not postponed, for every m in such a G_0, σ_i puts m into $\bigoplus_{S_i} U_j[s]$. Note that σ_i does not require any number in $\bigcup_{j<i} R_j$ to belong to any set unless that number already belongs to that set by virtue of $\vec{I}^*_i[s]$.

Lemma 5.8. *Suppose that $\vec{U}$ is constructed according to the prescriptions of σ_i in a construction in which σ_i operates within the programming environment described above. If $\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i)) = D_i$ then $\Gamma_i(\bigoplus_{S_i} \vec{I}_i(j) \oplus \bigoplus \mathcal{B}(Y_i)) = D_i$.*

Proof. Assume that $\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i)) = D_i$.

First note that if $n \in D_i$ then $\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i); n)\downarrow$. Since all of the sets in question are Σ^0_2, there is a stage s_0 such that $\Theta_i(\bigoplus_{S_i} U^*_j \oplus \bigoplus \mathcal{B}(Y_i)^*; n)[s]\downarrow$ during every stage s greater than or equal to s_0. But then there is a $*$-true stage s such that $\Theta_i(\bigoplus_{S_i} U^*_j \oplus \bigoplus \mathcal{B}(Y_i)^*; n)[s]\downarrow$. In such a stage, either we have that $\Gamma_i(\bigoplus_{S_i} \vec{I}_i(j)^* \oplus \bigoplus \mathcal{B}(Y_i)^*; n)[s]\downarrow$, or otherwise we force it to happen by enumerating an axiom $\langle n, \bigoplus_{S_i} \vec{I}^*_j[s] \oplus B_0\rangle$ into Γ_i (where $B_0 \subseteq \mathcal{B}(Y_i)^*[s]$). Since s is a true stage, $\bigoplus_{S_i} \vec{I}_i(j)^* \oplus \bigoplus \mathcal{B}(Y_i)^*[s]$ is contained in $\bigoplus_{S_i} \vec{I}_j \oplus \bigoplus \mathcal{B}(Y_i)$, and so the axiom in Γ_i applies to $\bigoplus_{S_i} \vec{I}_i(j) \oplus \bigoplus \mathcal{B}(Y_i)$. Consequently, if $n \in D_i$, then we conclude that $\Gamma_i(\bigoplus_{S_i} \vec{I}_i(j) \oplus \bigoplus \mathcal{B}(Y_i); n)\downarrow$.

Now, consider the converse. Suppose that $\Gamma_i(\bigoplus_{S_i} \vec{I}_i(j) \oplus \bigoplus \mathcal{B}(Y_i); n)\downarrow$ and let $\langle n, F\rangle$ be an axiom in Γ_i such that $F \subseteq \bigoplus_{S_i} \vec{I}_i(j) \oplus \bigoplus \mathcal{B}(Y_i)$. Let $\langle n, G_0 \oplus B_0\rangle$ be an axiom in Θ_i associated with $\langle n, F\rangle$. Since $\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i)) = D_i$, there is no w which is a witness to the inequality of $\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i))$ and D_i. The set $\bigoplus_{S_i} \vec{I}_i(j) \oplus \bigoplus \mathcal{B}(Y_i)$ is Σ^0_2 and n's not being a witness to the inequality is a Σ^0_2 property of n, and so there is an s_0 such that for all $s > s_0$, $w^*_i[s] > n$ and $F \subseteq \bigoplus_{S_i} \vec{I}_i(j)^* \oplus \bigoplus \mathcal{B}(Y_i)^*[s]$.

After stage s_0, the constraint that $G_0 \subseteq \bigoplus_{S_i} U_j$ for the sake of $\langle n, F\rangle$ can only be postponed if either there is a stage $s > s_0$ during which $w^*_i[s] \leq n$ or during which F does not appear to be consistent with $\bigoplus_{S_i} \vec{I}_i(j)^* \oplus \bigoplus \mathcal{B}(Y_i)^*$. By the choice of s_0, neither condition can apply. Thus, G_0 will be a subset of $\bigoplus_{S_i} U_j$. In addition, since F applies to $\bigoplus_{S_i} \vec{I}_i(j) \oplus \bigoplus \mathcal{B}(Y_i)$, B_0 is a subset of $\bigoplus \mathcal{B}(Y_i)$.

So $\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i); n)\downarrow$. Since $\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i)) = D_i$, we have that $n \in D_i$. Consequently, if $\Gamma_i(\bigoplus_{S_i} \vec{I}_i(j) \oplus \bigoplus \mathcal{B}(Y_i); n)\downarrow$ then $n \in D_i$, as required. □

We will prove that $\bigoplus_{S_i} \vec{I}_j \oplus \bigoplus \mathcal{B}(Y_i) \not\geq_e D_i$ and apply Lemma 5.8 to conclude that $\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i)) \neq D_i$.

Effect of the strategy when $\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i)) \neq D_i$. Consider the case in which $\Gamma_i(\bigoplus_{S_i} \vec{I}_i(j) \oplus \bigoplus \mathcal{B}(Y_i)) \neq D_i$ and w_i is the least witness to this inequality. There will be infinitely many stages during which $w^*_i[s] = w_i$, and only finitely many stages during which $w^*_i[s] < w_i$.

Our strategy may well enumerate infinitely many axioms into Γ_i and therefore infinitely many constraints on $\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i)$ for the sake of these axioms. Since some of these constraints may be for the sake of axioms in Γ_i with arguments less than w_i, we may impose constraints which are only postponed finitely often. We show that there are only finitely many of these.

During each of the infinitely many stages during which $w_i^*[s] = w_i$, any constraint associated with making $\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i))$ converge at an argument greater than or equal to w_i is postponed.

For the numbers n less than w_i, let s be large enough so that for all $t > s$ we have $w_i[t]^* \geq w_i$, and if $\Gamma_i(\bigoplus_{S_i} \vec{I}_i(j) \oplus \bigoplus \mathcal{B}(Y_i); n) \downarrow$, then there is an F_n contained in $\bigoplus_{S_i} \vec{I}_i(j) \oplus \bigoplus \mathcal{B}(Y_i)$ such that $\langle n, F_n\rangle$ in $\Gamma_i[s]$ and for all $t > s$, $F_n \subseteq \bigoplus_{S_i} \vec{I}_i(j)^* \oplus \bigoplus \mathcal{B}(Y_i)^*[t]$. For each of these n's and each $t > s$, we have that $\Gamma_i(\bigoplus_{S_i} \vec{I}_i(j)^* \oplus \bigoplus \mathcal{B}(Y_i)^*; n)[t]$ will converge. Consequently, σ_i will not enumerate any additional axioms into Γ_i with any of these n's as arguments. So there can be at most finitely many axioms for numbers less than w_i which apply to $\vec{I}_i$ in the limit. Constraints imposed for the sake of these axioms will only be postponed finitely often, and all other constraints will be postponed infinitely often, in particular they will be postponed during every $*$-true stage.

Case 2: Requirement $\Theta_i(\bigoplus \mathcal{B}(Y_i)) \neq X_i$ with $\mathcal{B}(y_i) \subseteq \mathcal{J}$ and $x_i \in R^{\mathcal{J}}$. Next, we consider the case when the oracle of the left term of the inequality is a join of sets in the image of $\mathcal{P}$.

If $\mathcal{B}(X_i) \not\subseteq \mathcal{BAB}(Y_i)$ then we may conclude that $\bigoplus \mathcal{B}(Y_i) \not\geq_e X_i$, once we prove that X_i is an upper bound for $\mathcal{B}(X_i)$. Since we are directly coding the elements of $\mathcal{B}(X_i)$ into X_i, the latter condition is immediate. If $\mathcal{B}(X_i) \not\subseteq \mathcal{BAB}(Y_i)$, then we say that $\Theta_i(\bigoplus \mathcal{B}(Y_i)) \neq X_i$ is a trivial requirement.

When $\mathcal{B}(X_i) \subseteq \mathcal{BAB}(Y_i)$, we implement a version σ_i of the Sacks coding strategy as follows. Since Condition (1) of Definition 5.1 applies and since $\mathcal{J} = \mathcal{BA}(x_i)$, we may fix an $h_i \in \mathcal{J} - \mathcal{BAB}(Y_i)$. Since $h_i \notin \mathcal{BAB}(Y_i)$, we have $\bigoplus \mathcal{B}(Y_i) \not\geq_e H_i$.

We enumerate an enumeration operator Δ_i and ensure that if $\Theta_i(\bigoplus \mathcal{B}(Y_i)) = X_i$, then $\Delta_i(X_i) = H_i$. With the same caveats as above, we let w_i be the least witness to the inequality between $\Theta_i(\bigoplus \mathcal{B}(Y_i))$ and X_i.

Sacks coding. We now adapt the coding strategy to enumeration reducibility.

Description. For each element of R_i which is not an element of $\vec{I}_i[s]$, we proceed as follows.

(1) If n is less than $w_i^*[s]$, then we set $X_i(m_n)[s] = H_i(n)[s]$ where m_n is the nth element of R_i.
(2) If n is greater than or equal to $w_i^*[s]$, we set $X_i(m_n)[s] = 0$.

The effect is to define X_i so that with finitely many exceptions, if $n < w_i$ then $X_i(m_n) = H(n)$, and if $n \geq w_i$ then $X_i(m_n) = 0$. Thus, we have the following lemma.

Lemma 5.9. *If X_i is enumerated according to the Sacks coding strategy, and if $\Theta_i(\bigoplus \mathcal{B}(Y_i)) = X_i$, then $X_i \geq_e H_i$.*

Since $\bigoplus \mathcal{B}(Y_i) \not\geq_e H_i$, we will later apply Lemma 5.9 to conclude $\Theta_i(\bigoplus \mathcal{B}(Y_i)) \neq X_i$.

Effect of the strategy when $\Theta_i(\bigoplus \mathcal{B}(Y_i)) \neq X_i$. By the choice of H_i, we have that $\Delta_i(\Theta(\bigoplus \mathcal{B}(Y_i)))$ cannot equal H_i. As we mentioned above, the effect is to define X_i so that except for the finite effect of $\vec{I}_i$, if $n < w_i$ then $X_i(m_n) = H(n)$, and if $n \geq w_i$ then $X_i(m_n) = 0$.

Our effect on X_i is then described by the finite set of numbers less than w_i which belong to H_i whose coding locations were not controlled by $\vec{I}_i$. Further, the only sets affected once we correct for the sake of coding are those sets which directly code the effects of incomparability strategies on X_i.

Case 3: Requirement $\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i)) \neq V_i$ with $\mathcal{B}(y_i) \subseteq \mathcal{J}$, $S_i \neq \emptyset$, and $v_i \in R^{\mathcal{J}}$. The third case for an incomparability strategy is the one in which both sides of the inequality include elements of $R^{\mathcal{J}}$. If $\mathcal{B}(V_i) \not\subseteq \mathcal{BAB}(Y_i)$, then we may conclude that $\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i) \not\geq_e V_i$ once we invoke the effects of strategies of Case 1 and prove that not all of the elements of $\mathcal{B}(V_i)$ are e-reducible to $\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i)$. In this case, we say that requirement $\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i)) \neq V_i$ is trivial.

Otherwise, our strategy is the simultaneous combination of our strategies in the previous two cases. We define H_i to be an element of $\mathcal{BA}(V_i) - \mathcal{BAB}(Y_i)$ and ensure the following two implications.

$$\Theta_i(\textstyle\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i)) = V_i \implies \Gamma_i(\bigoplus_{S_i} \vec{I}_i(j) \oplus \bigoplus \mathcal{B}(Y_i)) = V_i, \text{ and}$$
$$\Theta_i(\textstyle\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i)) = V_i \implies \Delta_i(V_i) = H_i$$

We use Sacks's preservation strategy for the first implication, regarding V_i as if it were in $\mathcal{P}$. We use Sacks's coding strategy for the second implication, now regarding V_i as a set under construction. We let their common value of w_i be the least witness to the inequality between $\Delta_i(\Gamma_i(\bigoplus_{S_i} \vec{I}_i(j) \oplus \bigoplus \mathcal{B}(Y_i)))$ and H_i.

Our analysis from the previous sections applies, both to show that these implications hold and to show that if $\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i)) \neq V_i$, then our strategy has a finitely described effect on $\bigoplus_{S_i} U_j$ and V_i.

Lemma 5.10. *Suppose that $\bigoplus_{S_i} U_j$ and V_i are approximated according to the above incomparability strategy in a construction in which σ_i operates within the programming environment described above. Then, if $\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i)) = V_i$ then $\Gamma_i(\bigoplus_{S_i} \vec{I}_i(j) \oplus \bigoplus \mathcal{B}(Y_i)) = V_i$ and $\Delta_i(V_i) = H_i$.*

5.4. Construction. We now describe our construction to combine strategies for all of the requirements.

Definition 5.11.

(1) Suppose r_i is the requirement $\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i)) \neq X_i$ and σ_i is our strategy to satisfy r_i. We write $\sigma_{i,N}$ and $\sigma_{i,\mathcal{P}}$ to refer to the Sacks preservation and Sacks coding components of σ_i, respectively. Depending on which of $S_i = \emptyset$ or $X_i \in \mathcal{P}$ holds, one or the other of $\sigma_{i,N}$ or $\sigma_{i,P}$ may be trivial.
(2) We write w_i to refer to the least witness to the inequality between X_i and $\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i))$ if there is one, and to refer to infinity otherwise.

The stage-by-stage construction. We organize our construction by stages, indexed by s. Each stage s is divided into $s + k$ many substages, indexed by i, where k is the number of comparability strategies.

During stage s, we are given stage s approximations to the elements in the image of $\mathcal{P}$. We define approximations $U_j[s]$ to the elements that we are constructing in the image of $R^{\mathcal{J}}$, approximations $\vec{I}_i[s]$ (for $i \geq k+1$) to the injury sets $\vec{I}_i$ defined below, and approximations $w_i[s]$. By the Recursion Theorem, we fix in advance a simultaneous true stage approximation to these quantities, and we use $X^*[s]$ and so forth to refer to the approximations to the above sets under the true stage approximation.

Comparability strategies. For each $u \in R^{\mathcal{J}}$, for each X in $\mathcal{Q}^{\mathcal{J}}$ such that $u > x$, we ensure that $X[s] \subseteq U[s]$.

Incomparability strategies. Fix a computable enumeration $(r_i : i > k)$ of all the nontrivial incomparability requirements. (We let $\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i)) \neq X_i$ denote the inequality required by r_i.)

We define $\vec{I}_i[s]$ at the beginning of substage i. For each i, $\vec{I}_i[s]$ will be a sequence of length $|R^{\mathcal{J}}|$ all of whose coordinates are finite sets.

For i less than or equal to k, σ_i is devoted to coding the elements of $\mathcal{P}$ into the appropriate U_j's. Since the coding strategies are not injured, we let $\vec{I}_i[s]$ be the sequence of empty sets.

We let $\vec{I}_{k+1}[s]$ be the sequence such that for each m less than $|R^{\mathcal{J}}|$, the mth coordinate of $\vec{I}_{k+1}[s]$ is the union of the elements of $\mathcal{B}(U_m)[s]$.

For the inductive step, suppose that $\vec{I}_i[s]$ is defined. Let $\vec{I}_{i+1}[s]$ be the sequence such that for all m less than to $|R^{\mathcal{J}}|$, the mth coordinate $\vec{I}_{i+1}(m)[s]$ of $\vec{I}_{i+1}[s]$ is the union of $\vec{I}_i(m)[s]$ with the set of numbers n such that during stage s or earlier, σ_i imposed a constraint which included n's belonging to $U_m[s]$ and that constraint was not postponed by σ_i during stage s.

We define $w_i[s]$ as in the section on approximating outcomes.

During substage i, we follow the instructions of the strategy described earlier for the requirement r_i, relative to the true stage approximations.

For each $u_m \in R^{\mathcal{J}}$, we define $U_m[s]$ once we have completed substage $s + k$ of stage s. We let $U_m[s]$ be the set of numbers n such that for some σ_i with i less than $s + k$, during stage s or earlier σ_i imposed a constraint which included n's belonging to $U_m[s]$ and that constraint was not postponed by σ_i during stage s.

5.5. Verification.

Lemma 5.12. *Let i be greater than k.*

(1) *The sets constructed satisfy all of the nontrivial inequality requirements r_i : $\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i)) \neq X_i$.*
(2) *$\vec{I}_i(j)$ is enumeration reducible to $\bigoplus \mathcal{B}(U_j)$ for each $j < |R^{\mathcal{J}}|$.*
(3) *$\vec{I}_i$ is uniformly enumeration reducible to $\bigoplus \mathcal{J}$. That is, the indices for the enumeration reductions are obtained as a computable function of i.*

Proof. Clearly, the second and third claims hold when i is equal to $k + 1$.

Suppose that clauses (ii) and (iii) of Lemma 5.12 hold for numbers greater than or equal to $k+1$ and less than or equal to i. We will show that clause (i) holds for i and clauses (ii) and (iii) hold for $i + 1$.

First, by applying the appropriate Lemma 5.8, 5.9, or 5.10, we may conclude that $\Theta_i(\bigoplus_{S_i} U_j \oplus \bigoplus \mathcal{B}(Y_i)) \neq X_i$. This verifies the clause (i) of Lemma 5.12. Consequently, there are only finitely many constraints which are imposed by σ_i and only postponed by it finitely often. Thus, for each $j < |R^{\mathcal{J}}|$, $\vec{I}_{i+1}(j) - \vec{I}_i(j)$ is finite and clause (ii) of the Lemma follows. Finally, each number n that appears in $\vec{I}_{i+1}(j)$ and not in $\vec{I}_i(j)$ does so as follows. Either an axiom applies to $\bigoplus_{S_i} \vec{I}_j \oplus \bigoplus \mathcal{B}(Y_i)$ (in the case of preservation strategies) or a number belongs to $H_i \in \mathcal{J}$ (in the case of coding strategies), and the least witness to the inequality is sufficiently large that the constraint on n is postponed at most finitely often by comparison with $w_i[s]$. By Lemma 5.7, the latter condition on postponement is enumeration reducible to $\bigoplus \mathcal{J}$ since all of the sets used as oracles in the equation to which w_i refers belong to $\mathcal{J}$. $\square$

We can now complete the proof of Proposition 5.5. Recall that this proposition has three clauses, which we verify in turn for the sets that we constructed above.

Clause (1) asserts that for all u in $R^{\mathcal{J}}$, $\bigoplus \mathcal{J} \geq_e U$. For each u and each n, $n \in U$ if and only if n is designated for U, n is constrained to belong to U by some strategy in the construction, and that constraint is postponed only finitely often. Equivalently, for each u and each n, $n \in U$ if and only if n is designated for U and there is an i such that n appears in $\vec{I}_i$. Since the sequences $\vec{I}_i$ are uniformly enumeration reducible to $\bigoplus \mathcal{J}$, U is enumeration reducible to $\bigoplus \mathcal{J}$.

Clause (2) asserts for all $y \in \mathcal{Q}$ and $d \in \mathcal{P}$, if $y \not\geq d$, then $\bigoplus \vec{U} \oplus \bigoplus \mathcal{B}(Y) \not\geq_e D$ where $\vec{u}$ is the set of elements $u \in R^{\mathcal{J}}$. If $\vec{u}$ is empty, then Clause 2 follows from the first extension condition of Definition 5.4, so we may assume that $\vec{u}$ is not empty. First consider the case when $\mathcal{B}(y) \subseteq \mathcal{J}$. If d is an element of $\mathcal{J}$, then for each Θ, we satisfied the requirement $\Theta(\bigoplus \vec{U} \oplus \bigoplus \mathcal{B}(Y)) \neq D$, and Clause 2 follows. If d is not an element of $\mathcal{J}$, then for any u in $R^{\mathcal{J}}$, $d \notin \mathcal{BA}(u)$. So let u be an element of $R^{\mathcal{J}}$ and let a be an element of $\mathcal{A}(u)$ such that $a \not\geq d$. But then $A \not\geq_e D$. Of course $A \geq_e \bigoplus \mathcal{J}$ and so $A \geq \bigoplus \vec{U} \oplus \bigoplus \mathcal{B}(Y)$. Now consider the case when $\mathcal{B}(y) \not\subseteq \mathcal{J}$. Then for each u in $\vec{u}$, $\mathcal{B}(y) \not\subseteq \mathcal{BA}(u)$ and so u is an element of $\mathcal{Z}(y)$. But then, the second extension condition ensures that $\mathcal{BA}(u) \subseteq \mathcal{B}(y)$, that is $\mathcal{J} \subseteq \mathcal{B}(y)$. Since $\vec{U}$ is enumeration reducible to $\bigoplus \mathcal{J}$, the condition $\bigoplus \vec{U} \oplus \bigoplus \mathcal{B}(Y) \not\geq_e D$ reduces to $\bigoplus \mathcal{B}(Y) \not\geq_e D$. As above, this latter condition is guaranteed again by the first extension condition.

Clause (3) asserts for all $y \in \mathcal{Q}$ and $v \in R^{\mathcal{J}}$, if $\vec{u}$ is the set of elements u in $R^{\mathcal{J}}$ such that $y \geq u$, and if $y \not\geq v$, then $\bigoplus \vec{U} \oplus \bigoplus \mathcal{B}(Y) \not\geq_e V$. As above, if $\mathcal{B}(y) \subseteq \mathcal{J}$, then we satisfied the requirements to ensure that $\bigoplus \vec{U} \oplus \bigoplus \mathcal{B}(Y) \not\geq_e V$. If $\mathcal{B}(y)$ is not a subset of $\mathcal{J}$ and $\vec{u}$ is not empty, then we get a contradiction: for u in $\vec{u}$, as above, the second extension condition implies that $\mathcal{BA}(u) \subseteq \mathcal{B}(y)$; the first extension condition of Definition 5.4 implies that there is an H in $\mathcal{BA}(V)$ such that $\bigoplus \mathcal{B}(Y) \not\geq_e H$; since v and u belong to $R^{\mathcal{J}}$, $\mathcal{BA}(V) = \mathcal{J} = \mathcal{BA}(U)$; and so, H is an element of $\mathcal{B}(Y)$ and not enumeration reducible to $\bigoplus \mathcal{B}(Y)$, contradiction. Finally, consider the case when $\vec{u}$ is empty and $\mathcal{B}(y)$ is not a subset of $\mathcal{J}$. Since $y \not\geq v$, the third extension condition applies, and either $\mathcal{B}(v) \not\subseteq \mathcal{B}(y)$ or $\mathcal{B}(y) \subseteq \mathcal{BA}(v)$. Since $\mathcal{B}(y) \not\subseteq \mathcal{J} = \mathcal{BA}(v)$, it must be the case that $\mathcal{B}(v) \not\subseteq \mathcal{B}(y)$. Let $h \in \mathcal{B}(v) - \mathcal{B}(y)$. By the action of our coding strategies, $V \geq_e H$ and, again by the first extension condition, $\bigoplus \mathcal{B}(Y) \not\geq_e V$ as required. $\square$

5.6. Completing the extension. Now we derive Theorem 5.2 from Proposition 5.5. We begin by showing that the sets we constructed in Proposition 5.5 satisfy the extension conditions of Definition 5.4.

Proposition 5.13. *Suppose that $\mathcal{P} \subseteq \mathcal{Q}$ are finite bounded partial orders, and that π is an embedding of $\mathcal{P}$ into the Σ^0_2-enumeration degrees such that $\mathcal{Q}$ satisfies the extension conditions over $\mathcal{P}$ and π. Suppose that $\mathcal{J}$ is a minimal extension ideal in $\mathcal{Q}$, and $\pi^{\mathcal{J}}$ is an embedding of $\mathcal{Q}^{\mathcal{J}}$ into the Σ^0_2-enumeration degrees extending π and satisfying the conditions of Proposition 5.5. Then $\mathcal{Q}$ satisfies the extension conditions over $\mathcal{Q}^{\mathcal{J}}$ and $\pi^{\mathcal{J}}$.*

Proof. We will be evaluating expressions such as $\mathcal{BA}(x)$ relative to $\mathcal{P}$ and also relative to $\mathcal{Q}^{\mathcal{J}}$. We use superscripts to specify our intention, such as $\mathcal{BA}(x)^{\mathcal{P}}$ and $\mathcal{BA}(x)^{\mathcal{Q}^{\mathcal{J}}}$.

It is immediate as argued in section 5.2 that $\pi^{\mathcal{J}}$ preserves comparability. To see that $\pi^{\mathcal{J}}$ preserves incomparability, let $y \in \mathcal{Q}^{\mathcal{J}}$ and observe that then $Y \equiv_e \bigoplus \vec{U} \oplus \bigoplus \mathcal{B}(Y)$ as defined in Proposition 5.5, and so Conditions 2 and 3 ensure the incomparability requirements involving y.

Condition 1. Suppose that $y \not\geq x$ in $\mathcal{Q}$. We must show that there is an h in $\mathcal{BA}(x)^{\mathcal{Q}^{\mathcal{J}}}$ such that $\bigoplus \mathcal{B}(Y)^{\mathcal{Q}^{\mathcal{J}}} \not\geq_e H$.

Case 1.1: $x \notin R^{\mathcal{J}}$. Since $\mathcal{Q}$ satisfies the extension conditions over $\mathcal{P}$ and π, let h be an element of $\mathcal{BA}(x)^{\mathcal{P}}$ such that $\bigoplus \mathcal{B}(Y)^{\mathcal{P}} \not\geq_e H$. Of course, since $\bigoplus \mathcal{B}(Y)^{\mathcal{P}} \not\geq_e H$, we have $y \not\geq h$ in $\mathcal{Q}$. By the second clause of Proposition 5.5, for $\vec{u}$ equal to the set of elements u in $R^{\mathcal{J}}$ such that $y \geq u$, $\bigoplus \vec{U} \oplus \bigoplus \mathcal{B}(Y)^{\mathcal{P}} \not\geq_e H$. But, $\bigoplus \vec{U} \oplus \bigoplus \mathcal{B}(Y)^{\mathcal{P}}$ is equal to $\bigoplus \mathcal{B}(Y)^{\mathcal{Q}^{\mathcal{J}}}$, and so the first extension condition is verified.

Case 1.2: $x \in R^{\mathcal{J}}$. The argument for this case is similar to the previous one, using the third clause of Proposition 5.5 in place of the second clause. Since $y \not\geq x$ in $\mathcal{Q}$, for $\vec{u}$ the set of elements u in $R^{\mathcal{J}}$ such that $y \geq u$, $\bigoplus \vec{U} \oplus \bigoplus \mathcal{B}(Y) \not\geq_e X$. Again, $\bigoplus \vec{U} \oplus \bigoplus \mathcal{B}(Y)$ is equal to $\bigoplus \mathcal{B}(Y)^{\mathcal{Q}^{\mathcal{J}}}$. Then x itself is an element of $\mathcal{BA}(x)^{\mathcal{Q}^{\mathcal{J}}}$ such that $\bigoplus \mathcal{B}(Y)^{\mathcal{Q}^{\mathcal{J}}} \not\geq_e X$.

Condition 2. We must verify that, for all $y \in \mathcal{Q} - \mathcal{Q}^{\mathcal{J}}$, then $\mathcal{Z}(y)^{\mathcal{Q}^{\mathcal{J}}} = \emptyset$ or $\mathcal{BA}(\mathcal{Z}(y) \cup \mathcal{B}(y))^{\mathcal{Q}^{\mathcal{J}}} \subseteq \mathcal{B}(y)^{\mathcal{Q}^{\mathcal{J}}}$, where $\mathcal{Z}(y)^{\mathcal{Q}^{\mathcal{J}}}$ is given by

$$\mathcal{Z}(y)^{\mathcal{Q}^{\mathcal{J}}} = \{z : z \in \mathcal{Q} - \mathcal{Q}^{\mathcal{J}},\ z < y,\ \text{and } \mathcal{B}(y)^{\mathcal{Q}^{\mathcal{J}}} \not\subseteq \mathcal{BA}(z)^{\mathcal{Q}^{\mathcal{J}}}\}.$$

Claim 5.14. *For all y in $\mathcal{Q} - \mathcal{Q}^{\mathcal{J}}$, $\mathcal{Z}(y)^{\mathcal{Q}^{\mathcal{J}}}$ is a subset of $\mathcal{Z}(y)^{\mathcal{P}}$.*

Proof. Suppose that y and z are in $\mathcal{Q} - \mathcal{Q}^{\mathcal{J}}$, and z is in $\mathcal{Z}(y)^{\mathcal{Q}^{\mathcal{J}}} - \mathcal{Z}(y)^{\mathcal{P}}$.

Since y and z belong to $\mathcal{Q} - \mathcal{Q}^{\mathcal{J}}$ and $R^{\mathcal{J}}$ is closed downward in $\mathcal{Q} - \mathcal{P}$, $\mathcal{A}(y)^{\mathcal{Q}^{\mathcal{J}}}$ and $\mathcal{A}(z)^{\mathcal{Q}^{\mathcal{J}}}$ are equal to $\mathcal{A}(y)^{\mathcal{P}}$ and $\mathcal{A}(z)^{\mathcal{P}}$, respectively. Consequently, $\mathcal{BA}(z)^{\mathcal{P}}$ is contained in $\mathcal{BA}(z)^{\mathcal{Q}^{\mathcal{J}}}$.

Since $z \notin \mathcal{Z}(y)^{\mathcal{P}}$, $\mathcal{B}(y)^{\mathcal{P}} \subseteq \mathcal{BA}(z)^{\mathcal{P}}$ and thus $\mathcal{B}(y)^{\mathcal{P}} \subseteq \mathcal{BA}(z)^{\mathcal{Q}^{\mathcal{J}}}$.

Since $z \in \mathcal{Z}(y)^{\mathcal{Q}^{\mathcal{J}}}$, $\mathcal{B}(y)^{\mathcal{Q}^{\mathcal{J}}} - \mathcal{BA}(z)^{\mathcal{Q}^{\mathcal{J}}}$ is not empty. Also, $\mathcal{B}(y)^{\mathcal{Q}^{\mathcal{J}}} - \mathcal{BA}(z)^{\mathcal{Q}^{\mathcal{J}}}$ is contained in $R^{\mathcal{J}}$, the set of u's in $\mathcal{Q} - \mathcal{P}$ such that $\mathcal{BA}(u)^{\mathcal{P}} = \mathcal{J}$. Let u be an element of $\mathcal{B}(y)^{\mathcal{Q}^{\mathcal{J}}} - \mathcal{BA}(z)^{\mathcal{Q}^{\mathcal{J}}}$.

We consider the cases depending on whether $u \in \mathcal{Z}(y)^{\mathcal{P}}$.

Suppose $u \in \mathcal{Z}(y)^{\mathcal{P}}$. Then the second extension condition for $\mathcal{Q}$ over $\mathcal{P}$ and π implies that $\mathcal{BA}(u)^{\mathcal{P}} \subseteq \mathcal{B}(y)^{\mathcal{P}}$. Of course, $\mathcal{BA}(u)^{\mathcal{P}} = \mathcal{J}$ and $U \leq_e \bigoplus \mathcal{J}$. But then, $U \leq_e \bigoplus \mathcal{B}(Y)^{\mathcal{P}}$. Since $\mathcal{B}(y)^{\mathcal{P}} \subseteq \mathcal{BA}(z)^{\mathcal{P}}$, for each a in $\mathcal{A}(z)^{\mathcal{P}}$, $A \geq_e \bigoplus \mathcal{B}(Y)^{\mathcal{P}}$. So, for all A in $\mathcal{A}(Z)^{\mathcal{P}} = \mathcal{A}(Z)^{\mathcal{Q}^{\mathcal{J}}}$, $A \geq_e U$. Consequently, $u \in \mathcal{BA}(z)^{\mathcal{Q}^{\mathcal{J}}}$, contradicting the choice of u.

Now, suppose u is not an element of $\mathcal{Z}(y)^{\mathcal{P}}$. Then $\mathcal{B}(y)^{\mathcal{P}} \subseteq \mathcal{BA}(u)^{\mathcal{P}}$. As in the previous paragraph, if $\mathcal{BA}(u)^{\mathcal{P}} \subseteq \mathcal{BA}(z)^{\mathcal{P}}$, then we obtain a contradiction. So, we may assume that $\mathcal{BA}(u)^{\mathcal{P}} \not\subseteq \mathcal{BA}(z)^{\mathcal{P}}$. But then, there must be an $a_z \in \mathcal{A}(z)^{\mathcal{P}}$ such that a_z is not above some element of $\mathcal{P}$ which is below every element of $\mathcal{P}$ which is above u. It follows that $a_z \not\geq u$. Now apply the third extension condition for $\mathcal{Q}$ over $\mathcal{P}$ and π to a_z and u: either $\mathcal{B}(u)^{\mathcal{P}} \not\subseteq \mathcal{B}(a_z)^{\mathcal{P}}$ or $\mathcal{B}(a_z)^{\mathcal{P}} \subseteq \mathcal{BA}(u)^{\mathcal{P}}$.

Since a_z belongs to $\mathcal{P}$ and $\mathcal{BA}(u)^{\mathcal{P}} = \mathcal{J}$, this condition reduces as follows: either a_z is not an upper bound for $\mathcal{B}(u)^{\mathcal{P}}$ or $a_z \in \mathcal{J}$. We have already concluded that $\mathcal{B}(y)^{\mathcal{P}} \subseteq \mathcal{BA}(z)^{\mathcal{P}}$, and we note that $u \leq y$ implies that $\mathcal{B}(u)^{\mathcal{P}} \subseteq \mathcal{B}(y)^{\mathcal{P}}$. Thus, a_z is an upper bound for $\mathcal{B}(u)^{\mathcal{P}}$. If a_z is an element of $\mathcal{J}$, then $\mathcal{BA}(z)^{\mathcal{P}} \subseteq \mathcal{J}$. By the minimality of $\mathcal{J}$, $\mathcal{BA}(z)^{\mathcal{P}} = \mathcal{J}$. But then $z \in \mathcal{Q}^{\mathcal{J}}$, a contradiction.

This finishes the proof of Claim 5.14. $\square$

Now, we complete the analysis of the second extension condition for $\mathcal{Q}$ over $\mathcal{Q}^{\mathcal{J}}$ and $\pi^{\mathcal{J}}$. Suppose that $\mathcal{Z}(y)^{\mathcal{Q}^{\mathcal{J}}}$ is not empty. We must show that $\mathcal{BA}(\mathcal{Z}(y) \cup \mathcal{B}(y))^{\mathcal{Q}^{\mathcal{J}}} \subseteq \mathcal{B}(y)^{\mathcal{Q}^{\mathcal{J}}}$. By Claim 5.14, We know that $\mathcal{Z}(y)^{\mathcal{P}}$ is not empty.

Case 2.1: First, we consider the case of b in $\mathcal{P} \cap \mathcal{BA}(\mathcal{Z}(y) \cup \mathcal{B}(y))^{\mathcal{Q}^{\mathcal{J}}}$. Suppose that b is not an element of $\mathcal{B}(y)^{\mathcal{Q}^{\mathcal{J}}}$. Since the second extension condition is satisfied for $\mathcal{Q}$ over $\mathcal{P}$ and π, $\mathcal{BA}(\mathcal{Z}(y) \cup \mathcal{B}(y))^{\mathcal{P}} \subseteq \mathcal{B}(y)^{\mathcal{P}}$. Then there must be an a such that a is an element of $\mathcal{A}(\mathcal{Z}(y) \cup \mathcal{B}(y))^{\mathcal{P}}$ and not in $\mathcal{A}(\mathcal{Z}(y) \cup \mathcal{B}(y))^{\mathcal{Q}^{\mathcal{J}}}$. By Claim 5.14, $\mathcal{Z}(y)^{\mathcal{Q}^{\mathcal{J}}} \subseteq \mathcal{Z}(y)^{\mathcal{P}}$, and so a is an upper bound for $\mathcal{Z}(y)^{\mathcal{Q}^{\mathcal{J}}}$. Therefore, we may fix $u \in R^{\mathcal{J}}$ such that $y \geq u$ and $a \not\geq u$. Apply the third extension condition for $\mathcal{Q}$ over $\mathcal{P}$ and π: either $\mathcal{B}(u)^{\mathcal{P}} \not\subseteq \mathcal{B}(a)^{\mathcal{P}}$ or $\mathcal{B}(a)^{\mathcal{P}} \subseteq \mathcal{BA}(u)^{\mathcal{P}}$. Since $a \in \mathcal{A}(\mathcal{Z}(y) \cup \mathcal{B}(y))^{\mathcal{P}}$, and $y \geq u$ implies $\mathcal{B}(u)^{\mathcal{P}} \subseteq \mathcal{B}(y)^{\mathcal{P}}$, we have $\mathcal{B}(u)^{\mathcal{P}} \subseteq \mathcal{B}(a)^{\mathcal{P}}$. So, $\mathcal{B}(a)^{\mathcal{P}} \subseteq \mathcal{BA}(u)^{\mathcal{P}}$, and consequently $a \in \mathcal{J}$. But then for every $z \in \mathcal{Q} - \mathcal{P}$, if $z < a$ then $\mathcal{BA}(z)^{\mathcal{P}} \subseteq \mathcal{J}$ and so $z \in \mathcal{Q}^{\mathcal{J}}$. Since $a \in \mathcal{A}(\mathcal{Z}(y))^{\mathcal{Q}^{\mathcal{J}}}$, $\mathcal{Z}(y)^{\mathcal{Q}^{\mathcal{J}}}$ must be empty, contradiction.

Case 2.2: We now consider the case of u in $R^{\mathcal{J}} \cap \mathcal{BA}(\mathcal{Z}(y) \cup \mathcal{B}(y))^{\mathcal{Q}^{\mathcal{J}}}$.

If u is below every a in $\mathcal{A}(\mathcal{Z}(y) \cup \mathcal{B}(y))^{\mathcal{P}}$, then $\mathcal{BA}(u)^{\mathcal{P}}$ is a subset of $\mathcal{BA}(\mathcal{Z}(y) \cup \mathcal{B}(y))^{\mathcal{P}}$. By the second extension condition for $\mathcal{Q}$ over $\mathcal{P}$ and π, $\mathcal{BA}(\mathcal{Z}(y) \cup \mathcal{B}(y))^{\mathcal{P}}$ is a subset of $\mathcal{B}(y)^{\mathcal{P}}$. Consequently, $\mathcal{BA}(u)^{\mathcal{P}}$, which is equal to $\mathcal{J}$, is a subset of $\mathcal{B}(y)^{\mathcal{P}}$. By the first clause of Proposition 5.5, $\bigoplus \mathcal{J}$ is an upper bound for U, so $U \leq_e \bigoplus \mathcal{B}(Y)^{\mathcal{P}}$. Consequently, by the third clause of Proposition 5.5, $y \geq u$ and so $u \in \mathcal{B}(y)^{\mathcal{Q}^{\mathcal{J}}}$, as desired.

Now, for the sake of a contradiction, we suppose that a is an element of $\mathcal{A}(\mathcal{Z}(y) \cup \mathcal{B}(y))^{\mathcal{P}}$ such that $a \not\geq u$. Apply the third extension condition for $\mathcal{Q}$ over $\mathcal{P}$ and π: either $\mathcal{B}(u)^{\mathcal{P}} \not\subseteq \mathcal{B}(a)^{\mathcal{P}}$ or $\mathcal{B}(a)^{\mathcal{P}} \subseteq \mathcal{BA}(u)^{\mathcal{P}}$. Look at the first case, $\mathcal{B}(u)^{\mathcal{P}} \not\subseteq \mathcal{B}(a)^{\mathcal{P}}$. Then there is a b such that $a \not\geq b$, $u \geq b$, and so b is in $\mathcal{P} \cap \mathcal{BA}(\mathcal{Z}(y) \cup \mathcal{B}(y))^{\mathcal{Q}^{\mathcal{J}}}$. In Case 2.1, we showed that $b \in \mathcal{P} \cap \mathcal{BA}(\mathcal{Z}(y) \cup \mathcal{B}(y))^{\mathcal{Q}^{\mathcal{J}}}$ implies that $b \in \mathcal{B}(y)^{\mathcal{P}}$, contradicting $a \in \mathcal{A}(\mathcal{Z}(y) \cup \mathcal{B}(y))^{\mathcal{P}}$ and $a \not\geq b$. Now, we look for a contradiction in the remaining case, $\mathcal{B}(a)^{\mathcal{P}} \subseteq \mathcal{BA}(u)^{\mathcal{P}}$. Since $u \in R^{\mathcal{J}}$, $\mathcal{BA}(u)^{\mathcal{P}} = \mathcal{J}$. Now, let $z \in \mathcal{Z}(y)^{\mathcal{Q}^{\mathcal{J}}}$, which is not empty by assumption. By Claim 5.14, $z \in \mathcal{Z}(y)^{\mathcal{P}}$ and so $\mathcal{BA}(z)^{\mathcal{P}} \subseteq \mathcal{B}(a)^{\mathcal{P}}$. Consequently, $\mathcal{BA}(z)^{\mathcal{P}} \subseteq \mathcal{B}(a)^{\mathcal{P}} \subseteq \mathcal{J}$, and by the minimality of $\mathcal{J}$ they are all equal. But then a is the greatest element of $\mathcal{J}$. By the first clause of Proposition 5.5, $\bigoplus \mathcal{J} \geq_e U$, hence $A \geq_e U$ and so $a \geq u$. This is a contradiction to the choice of u. The disjunction of two contradictions is a contradiction, and there can be no a as supposed.

Combining the analysis of the two components of $\mathcal{BA}(\mathcal{Z}(y) \cup \mathcal{B}(y))^{\mathcal{Q}^{\mathcal{J}}}$, we conclude that $\mathcal{BA}(\mathcal{Z}(y) \cup \mathcal{B}(y))^{\mathcal{Q}^{\mathcal{J}}} \subseteq \mathcal{B}(y)^{\mathcal{Q}^{\mathcal{J}}}$. This verifies the second extension condition for $\mathcal{Q}$ over $\mathcal{Q}^{\mathcal{J}}$ and $\pi^{\mathcal{J}}$.

Condition 3. We must verify that, for all y and x in $\mathcal{Q}$, if $y \not\geq x$ in $\mathcal{Q}$, then either $\mathcal{B}(x)^{\mathcal{Q}^{\mathcal{J}}} \not\subseteq \mathcal{B}(y)^{\mathcal{Q}^{\mathcal{J}}}$ or $\mathcal{B}(y)^{\mathcal{Q}^{\mathcal{J}}} \subseteq \mathcal{BA}(x)^{\mathcal{Q}^{\mathcal{J}}}$.

First, if $x \in \mathcal{Q}^{\mathcal{J}}$, then $x \in \mathcal{B}(x)^{\mathcal{Q}^{\mathcal{J}}} - \mathcal{B}(y)^{\mathcal{Q}^{\mathcal{J}}}$ and Condition 3 is satisfied. So, assume that $x \notin \mathcal{Q}^{\mathcal{J}}$. Further, note that since $R^{\mathcal{J}}$ is closed downward in $\mathcal{Q} - \mathcal{P}$

and $x \notin \mathcal{Q}^{\mathcal{J}}$, if $a \in \mathcal{Q}^{\mathcal{J}}$ and $a \geq x$, then $a \in \mathcal{P}$. Consequently, $\mathcal{A}(x)^{\mathcal{Q}^{\mathcal{J}}} = \mathcal{A}(x)^{\mathcal{P}}$.

Case 3.1: $y \in \mathcal{P}$. If $\mathcal{B}(x)^{\mathcal{Q}^{\mathcal{J}}} \not\subseteq \mathcal{B}(y)^{\mathcal{Q}^{\mathcal{J}}}$, then Condition 3 is satisfied. Assume that $\mathcal{B}(x)^{\mathcal{Q}^{\mathcal{J}}} \subseteq \mathcal{B}(y)^{\mathcal{Q}^{\mathcal{J}}}$, and consequently $\mathcal{B}(x)^{\mathcal{P}} \subseteq \mathcal{B}(y)^{\mathcal{P}}$. By the third extension condition for $\mathcal{Q}$ over $\mathcal{P}$ and π, since $y \in \mathcal{P}$ and $\mathcal{B}(y)^{\mathcal{P}} \subseteq \mathcal{B}\mathcal{A}(x)^{\mathcal{P}}$, we have $y \in \mathcal{B}\mathcal{A}(x)^{\mathcal{P}}$. Since, $\mathcal{A}(x)^{\mathcal{P}} = \mathcal{A}(x)^{\mathcal{Q}^{\mathcal{J}}}$, we may conclude that $y \in \mathcal{B}\mathcal{A}(x)^{\mathcal{Q}^{\mathcal{J}}}$ and so $\mathcal{B}(y)^{\mathcal{Q}^{\mathcal{J}}} \subseteq \mathcal{B}\mathcal{A}(x)^{\mathcal{Q}^{\mathcal{J}}}$.

Case 3.2: $y \in R^{\mathcal{J}}$. If $\mathcal{B}(x)^{\mathcal{P}} \not\subseteq \mathcal{B}(y)^{\mathcal{P}}$, then any element of $\mathcal{B}(x)^{\mathcal{P}} - \mathcal{B}(y)^{\mathcal{P}}$ is also an element of $\mathcal{B}(x)^{\mathcal{Q}^{\mathcal{J}}} - \mathcal{B}(y)^{\mathcal{Q}^{\mathcal{J}}}$, and Condition 3 is satisfied. Otherwise, by the third extension condition for $\mathcal{Q}$ over $\mathcal{P}$ and π, $\mathcal{B}(y)^{\mathcal{P}} \subseteq \mathcal{B}\mathcal{A}(x)^{\mathcal{P}}$.

Suppose that $\mathcal{B}(y)^{\mathcal{Q}^{\mathcal{J}}} \not\subseteq \mathcal{B}\mathcal{A}(x)^{\mathcal{Q}^{\mathcal{J}}}$. In particular, $y \notin \mathcal{B}\mathcal{A}(x)^{\mathcal{Q}^{\mathcal{J}}}$. Let a be an element of $\mathcal{A}(x)^{\mathcal{Q}^{\mathcal{J}}}$ such that $a \not\geq y$. Since, $\mathcal{A}(x)^{\mathcal{P}} = \mathcal{A}(x)^{\mathcal{Q}^{\mathcal{J}}}$, a is an element of $\mathcal{P}$. But then, Condition 3 applies to the pair a and y for $\mathcal{Q}$ over $\mathcal{P}$ and π: since $a \not\geq y$, either $\mathcal{B}(y)^{\mathcal{P}} \not\subseteq \mathcal{B}(a)^{\mathcal{P}}$ or $\mathcal{B}(a)^{\mathcal{P}} \subseteq \mathcal{B}\mathcal{A}(y)^{\mathcal{P}}$. But $\mathcal{B}(y)^{\mathcal{P}} \subseteq \mathcal{B}(a)^{\mathcal{P}}$ since we have assumed that $\mathcal{B}(y)^{\mathcal{P}} \subseteq \mathcal{B}\mathcal{A}(x)^{\mathcal{P}}$ and $a \in \mathcal{A}(x)^{\mathcal{P}}$, and so $\mathcal{B}\mathcal{A}(x)^{\mathcal{P}} \subseteq \mathcal{B}(a)^{\mathcal{P}}$. Consequently, $\mathcal{B}(a)^{\mathcal{P}} \subseteq \mathcal{B}\mathcal{A}(y)^{\mathcal{P}}$. But then $\mathcal{B}\mathcal{A}(x)^{\mathcal{P}} \subseteq \mathcal{B}\mathcal{A}(y)^{\mathcal{P}}$. Since $y \in \mathcal{Q}^{\mathcal{J}}$, $\mathcal{B}\mathcal{A}(y)^{\mathcal{P}} = \mathcal{J}$ and so $\mathcal{B}\mathcal{A}(x)^{\mathcal{P}} \subseteq \mathcal{J}$. By the minimality of $\mathcal{J}$, $\mathcal{B}\mathcal{A}(x)^{\mathcal{P}} = \mathcal{J}$ and so $x \in \mathcal{Q}^{\mathcal{J}}$, a contradiction since we have assumed that $x \notin \mathcal{Q}^{\mathcal{J}}$. Thus, $\mathcal{B}(y)^{\mathcal{Q}^{\mathcal{J}}} \subseteq \mathcal{B}\mathcal{A}(x)^{\mathcal{Q}^{\mathcal{J}}}$, and Condition 3 follows.

Case 3.3: $y \in \mathcal{Q} - \mathcal{Q}^{\mathcal{J}}$. If $\mathcal{B}(x)^{\mathcal{Q}^{\mathcal{J}}} \not\subseteq \mathcal{B}(y)^{\mathcal{Q}^{\mathcal{J}}}$, then Condition 3 is satisfied.

Otherwise, $\mathcal{B}(x)^{\mathcal{Q}^{\mathcal{J}}} \subseteq \mathcal{B}(y)^{\mathcal{Q}^{\mathcal{J}}}$, and hence $\mathcal{B}(x)^{\mathcal{P}} \subseteq \mathcal{B}(y)^{\mathcal{P}}$. By the third extension condition for $\mathcal{Q}$ over $\mathcal{P}$ and π, $\mathcal{B}(y)^{\mathcal{P}} \subseteq \mathcal{B}\mathcal{A}(x)^{\mathcal{P}}$.

If $\mathcal{B}(y)^{\mathcal{Q}^{\mathcal{J}}} = \mathcal{B}(y)^{\mathcal{P}}$, then since $\mathcal{A}(x)^{\mathcal{Q}^{\mathcal{J}}} = \mathcal{A}(x)^{\mathcal{P}}$, $\mathcal{B}(y)^{\mathcal{Q}^{\mathcal{J}}} \subseteq \mathcal{B}\mathcal{A}(x)^{\mathcal{Q}^{\mathcal{J}}}$, and Condition 3 is satisfied.

Now assume that $\mathcal{B}(y)^{\mathcal{Q}^{\mathcal{J}}} \neq \mathcal{B}(y)^{\mathcal{P}}$, and let $\vec{u}$ be the set of elements u in $R^{\mathcal{J}}$ such that $y \geq u$. Since for each u in $\vec{u}$, $\mathcal{B}(u)^{\mathcal{P}} \subseteq \mathcal{B}(y)^{\mathcal{P}}$ and $\mathcal{B}(y)^{\mathcal{P}} \subseteq \mathcal{B}\mathcal{A}(x)^{\mathcal{P}}$, we observe that for each u in $\vec{u}$, $\mathcal{B}(u)^{\mathcal{P}} \subseteq \mathcal{B}\mathcal{A}(x)^{\mathcal{P}}$.

Suppose that for some u in $\vec{u}$, $u \notin \mathcal{B}\mathcal{A}(x)^{\mathcal{Q}^{\mathcal{J}}}$. Fix $a \in \mathcal{A}(x)^{\mathcal{Q}^{\mathcal{J}}}$, and hence $a \in \mathcal{A}(x)^{\mathcal{P}}$, so that $a \not\geq u$. Since $\mathcal{Q}$ satisfies the third extension condition over $\mathcal{P}$ and π, either $\mathcal{B}(u)^{\mathcal{P}} \not\subseteq \mathcal{B}(a)^{\mathcal{P}}$ or $\mathcal{B}(a)^{\mathcal{P}} \subseteq \mathcal{B}\mathcal{A}(u)^{\mathcal{P}}$. Since $\mathcal{B}(u)^{\mathcal{P}} \subseteq \mathcal{B}\mathcal{A}(x)^{\mathcal{P}}$, the first disjunct is not possible. Consequently, $\mathcal{B}(a)^{\mathcal{P}} \subseteq \mathcal{B}\mathcal{A}(u)^{\mathcal{P}}$, and by evaluating these terms, $a \in \mathcal{J}$. But then $\mathcal{B}\mathcal{A}(x)^{\mathcal{P}} \subseteq \mathcal{J}$, and by the minimality of $\mathcal{J}$, $\mathcal{B}\mathcal{A}(x)^{\mathcal{P}} = \mathcal{J}$. Therefore, $x \in \mathcal{Q}^{J}$, a contradiction.

Thus, for every u in $\vec{u}$, $u \in \mathcal{B}\mathcal{A}(x)^{\mathcal{Q}^{\mathcal{J}}}$, and Condition 3 follows.

The three extension conditions are verified and we have completed the proof of Proposition 5.13. □

Now, we finish the proof of Theorem 5.2. Suppose that $\mathcal{P} \subseteq \mathcal{Q}$ are finite bounded partial orders such that $\mathcal{Q}$ satisfies all three of the extension conditions over $\mathcal{P}$. Let π be any embedding of $\mathcal{P}$ into the Σ^0_2-enumeration degrees preserving 0 and 1. Then note that $\mathcal{Q}$ satisfies all three of the extension conditions over $\mathcal{P}$ and π. We can decompose $\mathcal{Q}$ into a chain of partial orders $\mathcal{P} = \mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_k = \mathcal{Q}$ so that each $\mathcal{P}_{i+1}$ is obtained by adding the elements in a minimal extension ideal in $\mathcal{Q}$ over $\mathcal{P}_i$. Letting π_0 equal π, we can iteratively use Propositions 5.5 and 5.13 to extend π_i to an embedding π_{i+1} of $\mathcal{P}_{i+1}$ into the Σ^0_2-enumeration degrees so that $\mathcal{Q}$ meets the extension conditions over $\mathcal{P}_{i+1}$ and π_{i+1}. Then π_k is the extension of π to $\mathcal{Q}$ needed to verify Theorem 5.2. □

6. Concluding remarks. The next goal in this line of research would be to provide a decision procedure for the $\forall\exists$-theory of the Σ_2^0-enumeration degrees. We observe that the obstacles to extendibility of condition (1) in Theorem 1.5 can occur simultaneously, as we showed in section 2 above. However, by the following theorem of Ahmad, the obstacles of condition (3) cannot necessarily occur simultaneously:

Theorem 6.1 (Ahmad (see Ahmad, Lachlan [AL98])). *For any Σ_2^0-enumeration degrees $\mathbf{a}_0$ and $\mathbf{a}_1$, if*

$$\{\mathbf{x} \mid \mathbf{x} < \mathbf{a}_0\} = \{\mathbf{x} \mid \mathbf{x} < \mathbf{a}_1\}$$

then $\mathbf{a}_0 = \mathbf{a}_1$.

Now fix a partial ordering $\mathcal{P} = \{0, a_0, a_1, 1\}$ with a_0 and a_1 incomparable, as well as two extensions $\mathcal{Q}_i = \mathcal{P} \cup \{x_i\}$ (for $i \leq 1$) where $0 < x_i < a_i$ and $x_i \not\leq a_{1-i}$. Note then that Theorem 1.4 implies that $\mathcal{P}$ can be embedded into the Σ_2^0-enumeration degrees so as to prevent an extension to an embedding of $\mathcal{Q}_i$ for fixed i; but by Theorem 6.1, we cannot embed $\mathcal{P}$ into the Σ_2^0-enumeration degrees so as to simultaneously prevent an extension to an embedding of $\mathcal{Q}_i$ for both $i \leq 1$.

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