LOW UPPER BOUNDS OF IDEALS

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ABSTRACT. We show that there is a low T-upper bound for the class of K-trivial sets, namely those which are weak from the point of view of algorithmic randomness. This result is a special case of a more general characterization of ideals in Δ_2^0 T-degrees for which there is a low T-upper bound.

1. INTRODUCTION

1.1. **Background.** This paper is motivated by a question concerning the K-trivial sets, namely those sets which are computationally weak from the point of view of algorithmic randomness. The collection of K-trivial sets can be defined as consisting of exactly those sets for which prefix-free Kolmogorov complexity of initial segments grows as slowly as possible. However, there are at least three conceptually other ways to come to the same class, which is part of the interest in them.

Another part of the interest in this class lies in its properties when viewed as a subideal within the Δ_2^0 Turing degrees. Here, the K-trivial sets induce a Σ_3^0 ideal in the ω-r.e. T-degrees which is generated by its r.e. members, and the r.e. K-trivial sets induce a Σ_3^0 ideal in the r.e. T-degrees. This was proved by Nies [14] and partially also by Downey, Hirschfeldt, Nies, and Stephan [5] (see also [3] or [16]). Nies (unpublished, see [3]) also showed that there is a low$_2$ r.e. T-degree which is a T-upper bound for the class of K-trivial sets. However, Nies [15] also proved that there is no low r.e. T-upper bound for this class. Since all K-trivial sets are low, the latter result shows that the ideal is nonprincipal. Whether there is a low T-upper bound for the class of K-trivial sets remained unresolved. See, e.g., the list of open questions in Miller and Nies [12]).

This is the question which motivated this paper. We show that there is a low T-upper bound on the ideal of the K-trivial T-degrees. The proof applies more broadly, and we give a general characterization of those ideals in the Δ_2^0 T-degrees for which there is a low T-upper bound.

1.2. **Notation.** Our computability-theoretic notation generally follows Soare [22] and Odifreddi [17, 18]. An introduction to algorithmic randomness can be found in Li and Vitányi [11]. A short survey of it is also given in Ambos-Spies and Kučera

[1] and a longer one in Downey, Hirschfeldt, Nies, and Terwijn [4]. More recent progress is described in detail in forthcoming books of Downey and Hirschfeldt [3] and Nies [16].

We refer to the elements of 2^ω as sets or infinite binary sequences. We denote the collection of strings, i.e. finite initial segments of sets, by $2^{<\omega}$. The length of a string σ is denoted by $|\sigma|$, for a set X, we denote the string consisting of the first n bits of X by $X \upharpoonright n$. We let $\sigma * \tau$ denote the concatenation of σ and τ. We write $\sigma \preceq \tau$ to indicate that σ is a substring of τ, and similarly $\sigma \prec \tau$ to indicate that σ is a proper substring of τ. We further write $\sigma \prec X$ to indicate $X \upharpoonright |\sigma| = \sigma$. If $\sigma \in 2^{<\omega}$, then $[\sigma]$ denotes $\{X \in 2^\omega : \sigma \prec X\}$, and $Ext(\sigma) = \{\tau \in 2^{<\omega} : \sigma \preceq \tau\}$.

A Σ_1^0 class is a collection of sets that can be effectively enumerated. Such a class can be represented as $\bigcup_{\sigma \in W} [\sigma]$ for some (prefix-free) recursively enumerable (r.e.) set of strings W. The complements of Σ_1^0 classes are called Π_1^0 classes. Any Π_1^0 class can be represented by the class of all infinite paths through some recursive tree $\subseteq 2^{<\omega}$. We use also relativized versions, i.e. $\Pi_1^{0,X}$ classes.

A string σ is ω-extendable on a tree T if $\sigma \prec X$ for some infinite path X through T. A string σ is h-extendable on a tree T if there is a string $\tau \in T$ which extends σ, and $|\tau|$ is equal to $|\sigma| + h$. By ω-extendability of a string $\sigma \in 2^{<\omega}$ in a class $\mathcal{B} \subseteq 2^\omega$ we mean that there is a function $f \in \mathcal{B}$ which extends σ.

$[Tr]$ denotes the class of all infinite paths through a tree Tr.

Additionally to binary strings and trees $\subseteq 2^{<\omega}$ we will also use both finite sequences of elements of $\{0,1\} \times \{-1,0,1\}$ which we call p-strings and trees of p-strings (i.e. trees $\subseteq (\{0,1\} \times \{-1,0,1\})^{<\omega}$). We generalize notation for binary strings to p-strings in an obvious way.

K denotes prefix-free Kolmogorov complexity. We assume Martin-Löf's definition of 1-randomness as well as its relativization to an oracle.

Let f be a total function, i.e. $f \in \omega^\omega$, and let g be a (possibly) partial function, $g : \omega \mapsto \omega$. We say that f eventually dominates g if there is an n such that for all $x > n$, $g(x) \downarrow$ implies $f(x) \geq g(x)$.

Convention. If $\mathcal{C}$ is a class of sets, T-degrees of which form an ideal, we often speak of an ideal $\mathcal{C}$ by which we mean an ideal of T-degrees of members of $\mathcal{C}$.

2. PRELIMINARIES

We begin with basic definitions with which one can formulate the various characterizations of K-triviality. However, these characterizations differ in applicability as will be mentioned later.

Definition 2.1.   (1) $\mathcal{K}$ denotes the class of K-trivial sets, i.e. the class of sets A for which there is a constant c such that for all n, $K(A \upharpoonright n) \leq K(0^n) + c$.
   (2) $\mathcal{L}$ denotes the class of sets which are low for 1-randomness, i.e. sets A such that every 1-random set is also 1-random relative to A.
   (3) $\mathcal{M}$ denotes the class of sets that are low for K, i.e. the class of sets A for which there is a constant c such that for all σ, $K(\sigma) \leq K^A(\sigma) + c$.
   (4) A set A is a basis for 1-randomness if there is a Z such that $A \leq_T Z$ and Z is 1-random relative to A. The collection of such sets is denoted by $\mathcal{BR}$.

Nies [14] proved that $\mathcal{L} = \mathcal{M}$, Hirschfeldt and Nies, see [14], proved that $\mathcal{K} = \mathcal{M}$, and Hirschfeldt, Nies, and Stephan [6] proved that $\mathcal{BR} = \mathcal{K}$. Thus, all these

four classes are equal and we have, remarkably, four different characterizations of the same class.

Chaitin [2] proved that if a set is K-trivial then it is Δ_2^0. By a result of Kučera [9], low for 1-random sets are GL_1 and, thus, every K-trivial set is low. The lowness of these sets also follows from some recent results on this class of sets, see [14] or [4].

An interesting result shows that there is an effective listing of all K-trivial sets in the following way.

Theorem 2.1 (Downey, Hirschfeldt, Nies, and Stephan [5]). *There is an effective sequence $\{B_e, d_e\}_e$ of all the r.e. K-trivial sets and of constants such that each B_e is K-trivial via d_e.*

Nies [14] proved that the class of K-trivial sets is closed downwards under T-reducibility. Of course, the downward closure is immediate for the equality $\mathcal{K} = BR$, but that equality is a deeper fact. In the same paper, Nies also showed that for any K-trivial set A there is an r.e. K-trivial set B such that $A \leq_{tt} B$. Downey et al. [5] proved that the class of K-trivial sets is closed under join. Finally, Nies [14] showed that the K-trivial sets induce a Σ_3^0 ideal in the ω-r.e. T-degrees which is generated by its r.e. members, and r.e. K-trivial sets induce a Σ_3^0 ideal in the r.e. T-degrees.

On the other hand, the ideal is nonprincipal as the following theorem shows, since all K-trivials are low.

Theorem 2.2 (Nies [15]).
- *For each low r.e. set B, there is an r.e. K-trivial set A such that $A \nleq_T B$.*
- *For any effective listing $\{B_e, z_e\}_e$ of low r.e. sets and of their low indices there is an r.e. K-trivial set A such that $A \nleq_T B_e$ for all e.*

Remark. The proof uses a technique, known as Robinson low guessing method (introduced for low r.e. sets by Robinson [19]) which is compatible for low r.e. sets with a technique *do what is cheap*. Here *cheap* is defined in terms of a cost function (see, e.g. [3], [4], [16]). Alternatively, one could construct in the above Theorem a set which is low for 1-randomness instead of a K-trivial set and define *cheap* as having a small measure (see, e.g. Kučera and Terwijn [10] or Downey et al. [4]). However, the Robinson low guessing technique does not seem to generalize from r.e. sets to Δ_2^0 sets in a way which is compatible with the heuristic *do what is cheap*. In fact, it does not as Theorem 3.2 below shows.

An immediate corollary of Theorem 2.2 is that no low r.e. set can be a T-upper bound for the class $\mathcal{K}$.

Further, Theorem 2.2 was used by Nies [15, 14] and by Downey et al. [5] to show that four different characterizations of the same class, i.e. characterizations yielding $\mathcal{K}, \mathcal{L}, \mathcal{M}, \mathcal{BR}$ respectively, are not equally uniform. Especially, the uniformity in the characterization by K-triviality sets is weaker than the characterization by low for K. Unlike the constants by which a set is K-trivial, constants by which a set is low for K can be uniformly transformed into indices by which that set is low.

Theorem 2.3 (Nies [15], Nies [14], Downey, Hirschfeldt, Nies, and Stephan [5]).
- *There is no effective sequence $\{B_e, c_e\}_e$ of all the r.e. low for K sets with appropriate constants.*

- *There is no effective way to obtain from a pair (B, d), where B is an r.e. set that is K-trivial via d, a constant c such that B is low for K via c.*
- *There is no effective listing of all the r.e. K-trivial sets together with their low indices.*

There are several results on Σ_3^0 ideals of r.e. sets. One of the first such results is the following.

Theorem 2.4 (Yates [23])**.** *For any r.e set $A <_T \emptyset'$ the following conditions are equivalent.*

(1) $A'' \equiv_T \emptyset''$.
(2) $\{x : W_x \leq_T A\}$ *is a Σ_3^0 set.*
(3) *the class $\{W_x : W_x \leq_T A\}$ is uniformly r.e.*

On the other hand Nies (unpublished, see [3]) proved that for any proper Σ_3^0 ideal of r.e. sets there is a low_2 r.e. T-degree which is a T-upper bound for this ideal. Thus, together with the above theorem by Yates we have a characterization of ideals of r.e. sets for which there is a low_2 r.e. T-upper bound. An ideal has a low_2 T-upper bound if and only if it is a subideal of a proper Σ_3^0 ideal.

A characterization of ideals of r.e. sets (or ideals generated by their r.e. members) for which there is a low T-upper bound, not necessarily r.e., was open. We substantially use properties of $\{0, 1\}$-valued DNR functions (and their relativizations) in our construction of low T-upper bounds for ideals. We give a short review of basic properties of such functions here.

Definition 2.2. Let $\mathcal{PA}(B)$ denote the class of all $\{0, 1\}$-valued B-DNR functions, i.e. the class of functions $f \in 2^\omega$ such that $f(x) \neq \Phi_x(B)(x)$ for all x . If B is $\emptyset$ we simply speak of $\mathcal{PA}$.

Definition 2.3 (Simpson [21])**.** Write $\mathbf{b} << \mathbf{a}$ to mean that every infinite tree $T \subseteq 2^{<\omega}$ of T-degree $\leq \mathbf{b}$ has an infinite path of T-degree $\leq \mathbf{a}$.

Theorem 2.5 (Scott [20], Solovay (unpublished), see [21])**.** *The following conditions are equivalent:*

(1) $\mathbf{a}$ *is a T-degree of a $\{0, 1\}$-valued DNR function.*
(2) $\mathbf{a} >> \mathbf{0}$.
(3) $\mathbf{a}$ *is a T-degree of a complete extension of Peano arithmetic.*
(4) $\mathbf{a}$ *is a T-degree of a set separating some effectively inseparable pair of r.e. sets.*

Remark. By the implication from (1) to (2) in Theorem 2.5, $\mathcal{PA}$ is a "universal" Π_1^0 class. $\{0, 1\}$-valued DNR functions are also called PA sets and T-degrees $>> \mathbf{0}$ are also called PA degrees. Analogously, the class $\mathcal{PA}(B)$ is a "universal" $\Pi_1^{0,B}$ class. Simpson [21] proved that the partial ordering $<<$ is dense and $\mathbf{a} << \mathbf{b}$ implies $\mathbf{a} < \mathbf{b}$.

Definition 2.4. Let M be an infinite set and $\{m_i\}_i$ be an increasing list of all members of M. For $f \in 2^\omega$ by $Restr(f, M)$ we denote a function g defined for all i by $g(i) = f(m_i)$. Similarly, if $\mathcal{B} \subseteq 2^\omega$ then by $Restr(\mathcal{B}, M)$ we denote a class of functions $\{g : g = Restr(f, M) \ \& \ f \in \mathcal{B}\}$. Further, if $\sigma \in 2^{<\omega}$, then by $Restr(\sigma, M)$ we denote a string τ defined by $\tau(i) = \sigma(m_i)$ for all i such that $|m_i| < |\sigma|$.

Lemma 2.6. (1) *For every Π_1^0 class $\mathcal{B}$ which is a subclass of $\mathcal{PA}$ there is an infinite recursive set M such that if $\mathcal{B}$ is nonempty then $Restr(\mathcal{B}, M) = 2^\omega$, i.e. for every function $g \in 2^\omega$ there is a function $f \in \mathcal{B}$ such that $Restr(f, M) = g$. Moreover, an index of M can be found recursively from an index of $\mathcal{B}$.*

(2) *For every $\Pi_1^{0,B}$ class $\mathcal{B}$ which is a subclass of $\mathcal{PA}(B)$ there is an infinite recursive set M such that if $\mathcal{B}$ is nonempty then $Restr(\mathcal{B}, M) = 2^\omega$, where an index of M can be found uniformly-recursively from an index of $\mathcal{B}$, i.e. in a uniform way which does not depend on an oracle B.*

Proof. Lemma 2.6 is an application of Gödel's incompleteness phenomenon in the context of Π_1^0 classes of $\{0, 1\}$-valued B-DNR functions. Under a slight modification it was proved by Kučera [8]. For the convenience of the reader we first sketch the proof of the original version from [8], which is slightly technically easier. Then we show how to modify it to get the desired result stated in Lemma 2.6.

1. The original version from [8].

Let $\mathcal{G}_0$ denote the class of all $\{0, 1\}$-valued GNR functions, i.e. functions $f \in 2^\omega$ such that $f(<x, y>) \neq \varphi_x(y)$ for all x, y. Theorem 2 from [8] states the following. For every nonempty Π_1^0 class $\mathcal{B}$, $\mathcal{B} \subseteq \mathcal{G}_0$, there is x_0 such that for every set C there is a function $g \in \mathcal{B}$ such that $g(<x_0, y>) = C(y)$ for all y (and such x_0 can be found effectively from an index of $\mathcal{B}$).

Proof of this theorem.

Suppose $\mathcal{B}$ is a Π_1^0 subclass of $\mathcal{G}_0$. Observe that $\mathcal{B} \cap \{f \in 2^\omega : f(<x, j>) = \sigma(j), \ j < |\sigma|\} = \emptyset$ is a Σ_1^0 condition (with variables x and $\sigma \in 2^{<\omega}$). Thus, there is a total recursive function β such that $\varphi_{\beta(x)}(y)$ is defined for all y if and only if there is a string $\sigma \in 2^{<\omega}$ for which $\mathcal{B} \cap \{f \in 2^\omega : f(<x, j>) = \sigma(j), \ j < |\sigma|\} = \emptyset$, and if there is such a string, then the first such found under a standard search, say σ_0, is used to define $\varphi_{\beta(x)}(y) = 1 - \sigma_0(y)$ for all $y < |\sigma_0|$ and $\varphi_{\beta(x)}(y) = 0$ for all $y \geq |\sigma_0|$. By the recursion theorem there is x_0 such that $\varphi_{x_0} = \varphi_{\beta(x_0)}$.

It is easy to prove that if $\mathcal{B}$ is nonempty, then both $\mathcal{B} \cap \{f \in 2^\omega : f(<x_0, j>) = \sigma(j), \ j < |\sigma|\} \neq \emptyset$ for any $\sigma \in 2^{<\omega}$ and $\varphi_{x_0}(y)$ is not defined for any y. Really, it there were such a string σ for which $\mathcal{B} \cap \{f \in 2^\omega : f(<x_0, j>) = \sigma(j), \ j < |\sigma|\} = \emptyset$ and σ_0 were the first such found (as mentioned above) then every function $g \in \mathcal{G}_0$ would satisfy $g(<x_0, j>) = \sigma_0(j)$ for $j < |\sigma_0|$, which would immediately imply $\mathcal{B} = \emptyset$, a contradiction. Thus, for every string $\sigma \in 2^{<\omega}$ we have $\mathcal{B} \cap \{f \in 2^\omega : f(<x_0, j>) = \sigma(j), \ j < |\sigma|\} \neq \emptyset$. Using the compactness of the Cantor space 2^ω we have the required property, i.e. $\mathcal{B} \cap \{f \in 2^\omega : f(<x_0, y>) = C(y), \ y \in \omega\} \neq \emptyset$ for any set C.

2. Proof of the version stated in Lemma 2.6, part 1.

We easily modify the above proof by an additional use of the s-m-n theorem. Suppose a Π_1^0 class $\mathcal{B}$ is a subclass of $\mathcal{PA}$.

Let α be a partial recursive function such that $\alpha(z, y, w)$ is defined if and only if there is a string $\sigma \in 2^{<\omega}$ such that $\varphi_z(j)$ is defined for all $j < |\sigma|$ and $\mathcal{B} \cap \{f \in 2^\omega : f(\varphi_z(j)) = \sigma(j), \ j < |\sigma|\} = \emptyset$, and if such string exists then the first such found under a standard search, say σ_0, is used to define $\alpha(z, y, w) = 1 - \sigma_0(y)$ for all $y < |\sigma_0|$ and $\alpha(z, y, w) = 0$ for $y \geq |\sigma_0|$. By the s-m-n theorem there are total recursive functions γ and δ (even primitive recursive) such that $\alpha(z, y, w) = \varphi_{\gamma(z, y)}(w)$ and $\gamma(z, y) = \varphi_{\delta(z)}(y)$, for all z, y, w.

By the recursion theorem there is z_0 such that $\varphi_{z_0}(y) = \varphi_{\delta(z_0)}(y) = \gamma(z_0, y)$ for all

y. By properties of s-m-n functions we may assume that the function γ is increasing in both variables. Let h denote φ_{z_0}. Then h is a total increasing recursive function, and $\alpha(z_0, y, w) = \varphi_{h(y)}(w)$, for all y, w. A straightforward modification of the above proof of the original version shows that if $\mathcal{B}$ is nonempty then $\mathcal{B} \cap \{f \in 2^\omega : f(h(j)) = \sigma(j), \; j < |\sigma|\} \neq \emptyset$ for every string $\sigma \in 2^{<\omega}$. To verify that in details, suppose for a contradiction that there is a string $\sigma \in 2^{<\omega}$ for which $\mathcal{B} \cap \{f \in 2^\omega : f(h(j)) = \sigma(j), \; j < |\sigma|\} = \emptyset$ and let σ_0 denote the first such string found by a search realized to define values of the function α, namely values $\alpha(z_0, y, w)$. Then $\alpha(z_0, y, w)$ is defined and $\alpha(z_0, y, w) = \varphi_{h(y)}(w) = \varphi_{h(y)}(h(y))$ for all y, w, and, further, $\varphi_{h(y)}(h(y)) = 1 - \sigma_0(y)$ for all $y < |\sigma_0|$. Then, by the definition of the class $\mathcal{PA}$, i.e. the class of $\{0, 1\}$-valued DNR functions , every function $f \in \mathcal{PA}$ has to satisfy $f(h(y)) \neq \varphi_{h(y)}(h(y))$ and, thus, $f(h(y)) = \sigma_0(y)$ for all $y < |\sigma_0|$. This immediately yields that $\mathcal{B} = \mathcal{B} \cap \{f \in 2^\omega : f(h(j)) = \sigma_0(j), \; j < |\sigma_0|\}$ and, therefore, this proves that the class $\mathcal{B}$ is empty, a contradiction.

Thus, for every string $\sigma \in 2^{<\omega}$ we have $\mathcal{B} \cap \{f \in 2^\omega : f(h(j)) = \sigma(j), \; j < |\sigma|\} \neq \emptyset$. Using the compactness of the Cantor space 2^ω we have the required property, i.e. $\mathcal{B} \cap \{f \in 2^\omega : f(h(y)) = C(y), \; y \in \omega\} \neq \emptyset$ for any set C. Let M denote the recursive set which is the range of h. Obviously M has all required properties.

3. Part 2 of Lemma 2.6.

It is just a relativized version of part 1 and it is proved analogously.

$\square$

By Lemma 2.6, we can code arbitrary sets into members of $\Pi_1^{0,B}$ subclasses of $\mathcal{PA}(B)$. We illustrate it for an unrelativized case. Suppose that $\mathcal{B}$ is a nonempty Π_1^0 subclass of $\mathcal{PA}$ and M an infinite recursive set such that $Restr(\mathcal{B}, M) = 2^\omega$. Let a set C be given. If we take a class $\mathcal{E} = \{f : f \in \mathcal{B} \; \& \; Restr(f, M) = C\}$, then by our assumption $\mathcal{E}$ is nonempty. It is a $\Pi_1^{0,C}$ class and obviously any member B of $\mathcal{E}$ is T-above C. In a more general way, we may nest into a Π_1^0 subclass of $\mathcal{PA}$ not only a singleton $\{C\}$ as above, but even a given $\Pi_1^{0,C}$ class. Namely, with the above assumptions if $\mathcal{C}$ is a nonempty $\Pi_1^{0,C}$ class then a class $\mathcal{E} = \{f : f \in \mathcal{B} \; \& \; Restr(f, M) \in \mathcal{C}\}$ is nonempty, it is a $\Pi_1^{0,C}$ class and obviously any member of $\mathcal{E}$ is T-above some member of $\mathcal{C}$. A relativization of these tricks to $\Pi_1^{0,B}$ classes which are subclasses of $\mathcal{PA}(B)$ is straightforward. When combined with Low Basis Theorem of Jockusch and Soare [7] we easily get the following.

Example 2.1. For any low set A there is a low PA set B such that $A \leq_T B$. We may even require that additionally $\mathbf{a} << \mathbf{b}$ where $\mathbf{a}, \mathbf{b}$ are T-degrees of sets A, B respectively. A relativization of this fact to an oracle is straightforward.

3. Constructing low upper bounds for ideals

We show that there is a low T-upper bound for the class of K-trivial sets and give also a more general result about low T-upper bounds for ideals in the Δ_2^0 T-degrees.

Theorem 3.1. *Let $\mathcal{C}$ be a Σ_3^0 ideal of r.e. sets. Then the following conditions are equivalent.*

  (1) *There is a function F recursive in $\emptyset'$ which eventually dominates all partial functions recursive in any member of $\mathcal{C}$.*
  (2) *There is a low T-upper bound for $\mathcal{C}$.*

Theorem 3.1 follows from the next more general result.

Theorem 3.2. *Let $\mathcal{C}$ be an ideal in Δ_2^0 T-degrees. The following conditions (1) and (2) are equivalent.*

  (1) (a) *$\mathcal{C}$ is contained in an ideal $\mathcal{A}$ which is generated by a sequence of sets $\{A_n\}_n$ such that the sequence is uniformly recursive in $\emptyset'$ and*
        (b) *there is a function F recursive in $\emptyset'$ which eventually dominates any partial function recursive in any set with T-degree in $\mathcal{A}$.*
  (2) *There is a low T-upper bound for $\mathcal{C}$.*

Remark. We may equivalently require that a low T-upper bound (mentioned in Theorems 3.1 and 3.2) is PA since, as we saw, every low set has a low PA set T-above it. Thus, T-upper bounds which are PA are the most general case in this characterization.

Corollary 3.3. *There is a low set which is a T-upper bound for the class $\mathcal{K}$, i.e. for the ideal of K-trivial sets.*

Proof. As we already mentioned (see [14]) the class of r.e. K-trivial sets induces a Σ_3^0 ideal in the r.e. T-degrees, and the ideal $\mathcal{K}$ is induced by its r.e. members. Kučera and Terwijn [10] proved that there is a function F recursive in $\emptyset'$ which eventually dominates all partial functions recursive in any set which is low for 1-randomness. Since $\mathcal{L} = \mathcal{K}$, the corollary follows. $\square$

Remark. We explain the main obstacles of proving Theorem 3.2.

The implication from (2) to (1) is direct. If L is a low set, then $\emptyset'$ can compute the function $f : n \mapsto m$, where m is the strict supremum of the set

$$\{\{e\}^L(n) : e \leq n \ \& \ \{e\}^L(n) \text{ converges}\}.$$

This function eventually dominates every function recursive in L. Similarly, $\emptyset'$ can uniformly-recursively compute a sequence of sets consisting of exactly those sets which are recursive in L. For example, take the sequence X_e such that $X_e = \{n : \{e\}^L(n) = 1 \ \& \ \{e\}^L(j) \text{ converges for all } j \leq n\}$.

The implication from (1) to (2) is more subtle. Assume that $\mathcal{C}$ is generated by the uniformly recursive in $\emptyset'$ sequence of sets $\{A_n\}_n$ and there is a function F recursive in $\emptyset'$ which eventually dominates any partial function recursive in any set with T-degree in $\mathcal{C}$. (We identify $\mathcal{C}$ with the ideal $\mathcal{A}$ described in (1).)

We want to construct recursively in $\emptyset'$ a low set A for which $A \geq_T A_n$ for all n. The obstacle is that we do not have low indices of sets A_n, but we have to effectively decide facts about A'. A solution consists in using relativized Π_1^0 classes. Once we commit ourselves to A's satisfying a Π_1^0 sentence, we must ensure that sentence to be true in the limit. In other words, our commitment is that A should belong to the Π_1^0 class of reals for which the Π_1^0 sentence is true. Our next problem is coding the given sets A_n into the members of (relativized) Π_1^0 classes to which we have committed ourselves. Here a solution consists in using "rich" Π_1^0 classes or relativized Π_1^0 classes, like $\mathcal{PA}$ or $\mathcal{PA}(A_n)$. Finally, we come to the technical finesse in the construction. We use the given function F recursive in $\emptyset'$ which eventually dominates every partial function recursive in any A_n to replace missing low indices of A_n. More precisely, we replace questions about ω-extendability of a string on an A_n-recursive tree by questions about its finite-extendability where the depth to which extendability is required is computed by F. By a finite injury construction,

we can guarantee that eventually the answers to our questions about appropriate finite-extendability (of a string on a given A_n-recursive tree) computed by F are, in fact, correct answers to questions about ω-extendability.

Before presenting the whole (global) construction of the desired set A we first explain the main idea on a simple case, where we deal with just one given set A_n individually (in an isolated way). Later we describe how to combine all individual cases together to provide one global construction.

We give a simpler version for the case of K-trivial sets first and then give a more general version which is needed to prove Theorem 3.2.

Lemma 3.4. *There is a recursive procedure which given an index of an r.e. K-trivial set A produces a low set A^* and the lowness index for A^* such that $A \leq_T A^*$. That is, there are recursive functions f, g such that if W_e is K-trivial then $\Phi_{f(e)}(\emptyset')$ is a low set, $g(e)$ is its lowness index and $W_e \leq_T \Phi_{f(e)}(\emptyset')$.*

Remark. We do not claim that $A \leq_T A^*$ uniformly in an index of A. In fact, this uniformity would immediately provide uniformity for low indices of all r.e. K-trivial sets. But this contradicts the result of Nies [15] described in Theorem 2.2 or, equivalently, it contradicts Theorem 2.3. Also, by Theorem 2.2, the sets A^* mentioned in Lemma 3.4 cannot be obtained uniformly as r.e. sets (i.e. presented by their r.e. indices).

The previous lemma follows immediately from the more general result stated in Lemma 3.5.

Lemma 3.5. *Given a function F recursive in $\emptyset'$, there is a uniform way to obtain from a $\emptyset'$-index of a set A with the property that any partial function recursive in A is eventually dominated by F both a low set A^* and an index of lowness of A^* such that $A \leq_T A^*$. That is to say that there are recursive functions f, g such that if $\Phi_e(\emptyset')$ is total and equal to some set A so that any partial function recursive in A is eventually dominated by F then $\Phi_{f(e)}(\emptyset')$ is a low set, $g(e)$ is its lowness index and $A \leq_T \Phi_{f(e)}(\emptyset')$.*

Proof. Lemma 3.5 is the heart of the matter and its proof is the most technically involved section in our analysis.

The idea behind the proof is to combine forcing with Π_1^0 classes (as in the Jockusch and Soare [7] Low Basis Theorem) with coding sets into members of nonempty Π_1^0 subclasses of the class $\mathcal{PA}$. The given function F is used to approximate the answers to A'-questions. If A satisfies the given assumptions, our method will guarantee that the approximation will be correct from some point on. For the reader who is steeped in the priority methods of the recursively enumerable Turing degrees, our construction is the implementation of a Σ_2^0-strategy (like those in the Sacks Splitting Theorem) in which each action by the strategy restricts the construction to a yet smaller Π_1^0-class. Note, that $(A^*)'$ has to be uniformly recursive in $\emptyset'$. Thus, our $\emptyset'$-construction cannot change any decision about $(A^*)'(x)$ that it has already made.

We now describe our construction for one given A . So, let A be recursive in $\emptyset'$, let F be recursive in $\emptyset'$, and assume that any partial function recursive in A is eventually dominated by F.

Let PR denote $\{0, 1\} \times \{-1, 0, 1\}$ and let $PR^{<\omega}$ denote the set of all finite sequences of elements of PR. We call such sequences as p-strings. We use standard

notation associated with binary strings also for p-strings in an obvious way. Any p-string ρ may be viewed as a pair (σ, α) of a binary string $\sigma \in 2^{<\omega}$ and a sequence $\alpha \in \{-1, 0, 1\}^{<\omega}$ both of lengths equal to $|\rho|$, for which $\rho(j) = (\sigma(j), \alpha(j))$ for all $j < |\rho|$.

Further, for any finite sequence β from $\{-1, 0, 1\}^{<\omega}$ let $\beta^{\geq 0}$ denote a binary string α which arises by deleting all (-1)'s from β. Similarly, for any infinite sequence X from $\{-1, 0, 1\}^{\omega}$, $X^{\geq 0}$ denote (finite or infinite) binary sequence arising by deleting all (-1)'s from X.

Working recursively in $\emptyset'$, we construct an infinite perfect tree PT, a subtree of $PR^{<\omega}$. An infinite path P in PT consists of a set X in which A is recursive and a coding of X'. Since $\emptyset'$ can compute a path in PT uniformly, $\emptyset'$ can uniformly compute a low set T-above A together with the lowness index for that set.

Trying to keep our presentation simple, we view the function F as defined not on ω but on $PR^{<\omega}$, i.e. on p-strings. Define

$$F^*(k) = max\{F(\rho) : \rho \in PR^{<\omega} \ \& \ |\rho| = k\}.$$

Let G be a total recursive function such that the sequence $\{G(x, s)\}_s$ has the limit $F^*(x)$, for all x. By replacing functions F^* and G with possibly larger functions we may assume the following:

- the sequence $\{G(x, s)\}_s$ is nondecreasing
- $F^*(x)$ is greater or equal than the modulus of this limit, for all x, i.e. $j \geq F^*(x)$ implies $G(x, j) = F^*(x)$ for all j, x
- the function F^* is increasing.

With any p-string $\rho = (\sigma, \alpha)$ we will effectively associate a recursive tree $Tr_\rho \subseteq 2^{<\omega}$ (see below). We may assume, without loss of generality, that F^* is growing sufficiently fast so that if such tree is finite then the value of $F^*(|\rho|)$ is greater than the maximal d such that σ is d-extendable on this tree Tr_ρ. That is to say that if Tr_ρ is a finite tree, then $F^*(|\rho|)$ is at least as large as the height of Tr_ρ above σ.

We now build, recursively in $\emptyset'$, an infinite perfect tree PT of p-strings. We ensure that for any infinite path $Z = (X, Y)$ on PT, $X \in 2^{\omega}, Y \in \{-1, 0, 1\}^{\omega}$, and if Z is recursive in $\emptyset'$, then

$$X \text{ is low}, Y^{\geq 0} = X' \text{ and } A \leq_T X.$$

We will build tree PT inductively by stages. Let S_0 consists of the empty p-string (denoted by Λ). At stage $e > 0$, we will produce a finite collection of p-strings S_e by extending p-strings from S_{e-1} and we will restrict our tree PT, at stage e, to those p-strings compatible with p-strings from S_e. Thus, we define a sequence of finite trees, ordered under end-extension, and let PT be their union.

At stage $e + 1$, we will simultaneously and continuously in PT decide the e-th instance of the jump operator on the first coordinates X of the infinite paths (X, Y) through PT.

We fix some notation. Let DTr_e, the divergence tree for Φ_e, denote the recursive tree

$$DTr_e = \{\sigma \in 2^{<\omega} : \Phi_{e, |\sigma|}(\sigma)(e) \uparrow\}$$

and let $\mathcal{D}_e = [DTr_e]$ be the set of functions f such that $\Phi_e(f)(e) \uparrow$.

Finally, let $TrPA(A)$ denote an A-recursive tree $\subseteq 2^{<\omega}$ such that $[TrPA(A)] = \mathcal{PA}(A)$. That is to say, the class of infinite paths through $TrPA(A)$ is the class of all

$\{0,1\}$-valued A-DNR functions. Given an infinite recursive set M, let $CdTr(A,M)$ denote the subtree of $2^{<\omega}$ recursive in A defined by

$$CdTr(A,M) = \{\sigma \in 2^{<\omega} : Restr(\sigma,M) \in TrPA(A)\}.$$

Thus,

$$[CdTr(A,M)] = \{f : Restr(f,M) \in \mathcal{PA}(A)\}.$$

Here, $CdTr$ stands for a coding tree. Observe, that any member of $[CdTr(A,M)]$ is T-above A. This is our way of coding A into any X such that, for some Y, (X,Y) is an infinite path through the tree PT.

With each p-string $\rho = (\sigma,\alpha)$, we effectively associate a recursive tree $Tr_\rho \subseteq 2^{<\omega}$ and a Π^0_1 class $\mathcal{B}_\rho = [Tr_\rho]$ in the following way. Let first Tr_Λ denote a recursive tree such that $[Tr_\Lambda]$ equals $\mathcal{PA}$. Further, if $\beta = \alpha^{\geq 0}$, then $Tr_{(\sigma,\alpha)} = (Tr_\Lambda \cap Ext(\sigma)) \bigcap_{\beta(j)=0} DTr_j$. Intuitively, $\mathcal{B}_{(\sigma,\alpha)} = [Tr_{(\sigma,\alpha)}]$ is a restriction of $\mathcal{PA} \cap [\sigma]$ to a Π^0_1 class of sets X for which $j \notin X'$ for any j for which $\beta(j) = 0$.

We will ensure that for any p-string $(\sigma,\alpha) \in S_{e+1}$ one of the following conditions holds.

- $e \in X'$ for every $X \in [Tr_{(\sigma,\alpha)}]$
- $e \notin X'$ for every $X \in [Tr_{(\sigma,\alpha)}]$

In addition, with each $\rho \in PT$ we associate (recursively in $\emptyset'$) an infinite recursive set M_ρ. Each set M_ρ represents a way of coding of A into (first coordinate of) infinite paths extending ρ through PT. We will prove that along any infinite path through PT there will be only finitely many changes in the set that is so associated. In other words, along each path (X,Y) our coding of A into X will stabilize and ensure that $X \geq_T A$.

In outline, we begin by letting M_Λ be an infinite recursive set such that $Restr(\mathcal{PA}, M_\Lambda) = 2^\omega$. In other words, we commit ourselves to building a PA set (i.e. a $\{0,1\}$-valued DNR function) and we fix an infinite set M_Λ for coding A. At stage $e+1$ in our construction, if $\rho = (\sigma,\alpha) \in PR \cap S_e$ and we can extend ρ during stage $e+1$ without injury (as described below), then we will associate the same infinite recursive set M_ρ with the extensions of ρ that we add in S_{e+1}. Otherwise, our construction may be injured at ρ. In this case, in order to fulfill our commitments about X and X' which are specified by ρ, we must abandon the set M_ρ as the place to code A. We then specify new infinite recursive sets M^+. The technical device of the construction is to maintain the ability to keep numbers out of X' as specified by ρ as we monitor the coding of A into the extensions of ρ in PT.

With each p-string ρ and infinite recursive set M, we define the A-recursive tree

$$Tr_\rho(A,M) = Tr_\rho \cap CdTr(A,M).$$

Similarly, we let

$$\begin{aligned} \mathcal{B}_\rho(A,M) &= [Tr_\rho(A,M)] \\ &= \mathcal{B}_\rho \cap \{f : Restr(f,M) \in \mathcal{PA}(A)\} \end{aligned}$$

Intuitively, if $\rho = (\sigma,\alpha)$, then $\mathcal{B}_\rho(A,M)$ is a restriction of $\mathcal{PA} \cap [\sigma]$: first to the class of sets X for which $j \notin X'$ for any j for which $\beta(j) = 0$ and, second, to the class of sets X for which $Restr(X,M) \in \mathcal{PA}(A)$.

For any p-string (σ,α) which is on our tree PT, we will ensure the following two properties.

- ω-extendability of σ on $Tr_{(\sigma,\alpha)}$ (i.e. in $\mathcal{B}_{(\sigma,\alpha)}$)

- $F^*(|(\sigma, \alpha)|)$-extendability of σ on $Tr_{(\sigma, \alpha)}(A, M_{(\sigma, \alpha)})$

As already indicated earlier, we suppose that F^* grows sufficiently fast so that $Tr_{(\sigma, \alpha)}$ is finite if and only if σ is not $F^*(|(\sigma, \alpha)|)$-extendable on tree $Tr_{(\sigma, \alpha)}$.

To help the reader we give here a concentrated list of notations most frequently used in what follows.

PR denotes $\{0, 1\} \times \{-1, 0, 1\}$, $PR^{<\omega}$ denotes the set of all finite sequences of elements of PR, i.e. the set of p-strings.

PT denotes an infinite perfect tree recursive in $\emptyset'$, a subtree of $PR^{<\omega}$, (to be built).

DTr_e denotes the divergence tree for Φ_e, i.e. the recursive tree $DTr_e = \{\sigma \in 2^{<\omega} : \Phi_{e,|\sigma|}(\sigma)(e) \uparrow\}$, and $\mathcal{D}_e = [DTr_e]$ denotes the set of functions f such that $\Phi_e(f)(e) \uparrow$.

$TrPA(A)$ denotes an A-recursive tree $\subseteq 2^{<\omega}$ such that $[TrPA(A)] = \mathcal{PA}(A)$, i.e. the class of all infinite paths through $TrPA(A)$ is the class of all $\{0, 1\}$-valued A-DNR functions.

$CdTr(A, M)$, for an infinite recursive set M, (a coding tree), denotes the subtree of $2^{<\omega}$ recursive in A defined by $CdTr(A, M) = \{\sigma \in 2^{<\omega} : Restr(\sigma, M) \in TrPA(A)\}$. Thus, the class of all infinite paths through this tree is just the class of all functions $f \in 2^\omega$ for which $Restr(f, M) \in \mathcal{PA}(A)$ (i.e. functions coding in a recursive projection given by M a function from $\mathcal{PA}(A)$).

Tr_ρ, for each p-string $\rho = (\sigma, \alpha)$, denotes a recursive subtree of $2^{<\omega}$ and $\mathcal{B}_\rho$ denotes a Π^0_1 class $\mathcal{B}_\rho = [Tr_\rho]$ such that $Tr_{(\sigma, \alpha)} = (Tr_\Lambda \cap Ext(\sigma)) \bigcap_{\beta(j)=0} DTr_j$, where Tr_Λ is a recursive tree with $[Tr_\Lambda] = \mathcal{PA}$ and $\beta = \alpha^{\geq 0}$. I.e., $\mathcal{B}_{(\sigma, \alpha)}$ is equal to $\mathcal{PA} \cap [\sigma] \cap \{X : \beta(j) = 0 \rightarrow j \notin X'\}$.

For a p-string ρ and an infinite recursive set M, $Tr_\rho(A, M)$ denotes the A-recursive tree $Tr_\rho \cap CdTr(A, M)$ and $\mathcal{B}_\rho(A, M)$ denotes $[Tr_\rho(A, M)]$, i.e. $\mathcal{B}_\rho(A, M)$ is equal to $\mathcal{B}_\rho \cap \{f : Restr(f, M) \in \mathcal{PA}(A)\}$. Intuitively, if $\rho = (\sigma, \alpha)$, then $\mathcal{B}_\rho(A, M)$ is a restriction of $\mathcal{PA} \cap [\sigma]$: first to the class of sets X for which $j \notin X'$ for any j for which $\beta(j) = 0$ and, second, to the class of sets X for which $Restr(X, M) \in \mathcal{PA}(A)$.

Now, we present the precise recursion step of our construction.

3.1. **Stage** $e + 1$. Let $\rho = (\sigma, \alpha)$ be a p-string from S_e. We consider several cases.

Case 1. At least one of $\sigma * j$, for $j = 0, 1$, is both

- ω-extendable on $Tr_\rho \cap DTr_e$ (i.e. in $\mathcal{B}_\rho \cap \mathcal{D}_e$)
- and $F^*(|\rho| + 1)$-extendable on $Tr_\rho(A, M_\rho) \cap DTr_e$.

Then for all such j's put $(\sigma * j, \alpha * 0)$ into S_{e+1} and let $M_{(\sigma * j, \alpha * 0)}$ be M_ρ . In this case, we have ensured that $e \notin X'$, without injury.

Case 2. The previous case does not apply.

First observe, that necessarily $\mathcal{B}_\rho(A, M_\rho) \cap \mathcal{D}_e$ is empty (otherwise we would have the previous case). Further, this condition is recursively recognized relative to $\emptyset'$. However, since we are not working with a low index of A, we cannot determine recursively in $\emptyset'$ whether $\mathcal{B}_\rho(A, M_\rho)$ is empty. We consider two subcases. Either it is possible to ensure $e \in X'$ without injury at this case or we detect an injury. Injuries will be explained in detail below.

If we take for each $j = 0, 1$ all strings $\tau, \tau \succeq \sigma * j$, of length $|\rho| + 1 + F^*(|\rho| + 1)$, then the only such strings which are on $Tr_\rho(A, M_\rho)$ and are ω-extendable in $\mathcal{B}_\rho$ (if there are such at all) are not ω-extendable on DTr_e, (i.e. $[\tau] \cap \mathcal{D}_e$ is empty). So,

for $j = 0, 1$ take all $\gamma, \gamma \succeq \sigma * j$ (if such exist at all), which satisfy the following conditions.

- $\gamma \in Tr_\rho(A, M_\rho)$
- γ is ω-extendable on Tr_ρ (i.e. extends to an element of $\mathcal{B}_\rho$)
- γ has length $\leq |\rho| + 1 + F^*(|\rho| + 1)$
- $[\gamma] \cap \mathcal{D}_e$ is empty

Now we split into two subcases, depending on whether there is a string γ as above which is sufficiently extendable on $Tr_\rho(A, M_\rho)$.

Subcase 2.a. There are a string γ and j, such that $0 \leq j \leq 1$, $\gamma \succeq \sigma * j$, γ is both $F^*(|\gamma|)$-extendable on $Tr_\rho(A, M_\rho)$ and ω-extendable on Tr_ρ (i.e. in $\mathcal{B}_\rho$), $[\gamma] \cap \mathcal{D}_e$ is empty and $|\gamma| \leq |\rho| + 1 + F^*(|\rho| + 1)$.

Then for any such γ, j for which no $\tau, \sigma * j \preceq \tau \prec \gamma$ has this property, put $(\gamma, \alpha * (-1)^k * 1)$ into S_{e+1}, where $k = |\gamma| - |\sigma * j|$ $(= |\gamma| - |\rho| - 1)$, and let also $M_\tau = M_\rho$ for any $\tau, \rho \prec \tau \preceq (\gamma, \alpha * (-1)^k * 1)$.

In this case, we have ensured that $e \in X'$, without injury.

Subcase 2.b. Now assume that neither of the two previous situations applies. Then, we are unable to respect our commitments to deciding the jump while continuing the coding of A. This is the case in which we injure our coding strategy.

Observe, that $\mathcal{B}_\rho(A, M_\rho)$ must be empty. This means that σ is only finitely-extendable on $Tr_\rho(A, M_\rho)$. Further, from ρ and using A, we can compute an upper bound of this finite-extendability. Thus, there are strings γ, $\gamma \succeq \sigma$, with the following properties.

- γ is ω-extendable on Tr_ρ (i.e. in $\mathcal{B}_\rho$)
- $F^*(|\gamma|)$-extendable on $Tr_\rho(A, M_\rho)$

but, however, neither of the immediate extensions of γ is both ω-extendable on Tr_ρ (i.e. in $\mathcal{B}_\rho$) and $F^*(|\gamma|+1)$-extendable on $Tr_\rho(A, M_\rho)$. There are only finitely many of these strings γ. In particular, each is less than or equal to $|\rho| + 1 + F^*(|\rho| + 1)$. Finally, note that $Tr_\rho(A, M_\rho)$ is a subtree of Tr_ρ.

Then for each such γ, we say that an injury occurred at $(\gamma, \alpha * (-1)^k)$, where $k = |\gamma| - |\sigma|$ $(= |\gamma| - |\rho|)$, and we do the following.

Let for $j = 0, 1$, d_j denote the maximal d such that $\gamma * j$ is d-extendable on $Tr_\rho(A, M_\rho)$. Since $d_j < F^*(|\gamma| + 1)$, let t_0 be the least t such that $G(|\gamma| + 1, t) > d_j$ for both $j = 0, 1$. Intuitively, for strings γ in the current situation, at step t_0 the recursive approximation of F^* by G is able to see that an injury occurred.

We also know that the maximal d for which γ is d-extendable on $Tr_\rho(A, M_\rho)$ is greater than $F^*(|\gamma|)$. Denote it as $d_\emptyset$. We now position ourselves to use the hypothesis that F eventually dominates every function which is partial recursive relative to A. We will use $d_\emptyset$ to define a value of an A-partial recursive function at input $(\gamma, \alpha * (-1)^k)$, which is greater than the corresponding value of F^*. Precisely, the value of the defined function will be greater than the value $F^*(|\gamma|)$ and, therefore, also greater than $F(\gamma, \alpha * (-1)^k)$, since $F^*(|\rho|) \geq F(\rho)$ for all p-strings ρ.

To summarize, $d_\emptyset > F^*(|\gamma|)$, but $d_j < F^*(|\gamma| + 1)$, $j = 0, 1$ and also $d_\emptyset = 1 + max(d_0, d_1)$. Further, there is at least one string $\tau, \tau \succ \gamma$, which is ω-extendable on $Tr_{(\gamma, \alpha * (-1)^k)}$ (i.e. in $\mathcal{B}_{(\gamma, \alpha * (-1)^k)}$) and for which $|\tau| - |\gamma| = t_0$. Our ongoing commitments concerning the jumps of the paths in PT can be enforced on the extensions of these strings τ.

For each such τ, we have one of the following two possibilities.

(1) The first possibility is that $[\tau] \cap \mathcal{D}_e = \emptyset$. Then for $q = k + t_0 = |\tau| - |\sigma|$ denote $(\tau, \alpha * (-1)^{q-1} * 1)$ by ξ, put ξ into S_{e+1}, and let $M_\eta = M_\rho$ for any p-string η, $\rho \prec \eta \prec \xi$. Finally, apply Lemma 2.6 to effectively find an infinite recursive set M^+ such that $Restr(\mathcal{B}_\xi, M^+) = 2^\omega$ and let $M_\xi = M^+$.

(2) The second possibility is that $[\tau] \cap \mathcal{D}_e \neq \emptyset$. Then for $q = k + t_0 = |\tau| - |\sigma|$ denote $(\tau, \alpha * (-1)^{q-1} * 0)$ by ξ, put ξ into S_{e+1}, and let $M_\eta = M_\rho$ for any p-string η, $\rho \prec \eta \prec \xi$. Again, apply Lemma 2.6 to effectively find an infinite recursive set M^+ such that $Restr(\mathcal{B}_\xi, M^+) = 2^\omega$ and let $M_\xi = M^+$.

In both of these possibilities, we start with a new version of coding of A into (first coordinate of) infinite paths through PT extending ξ.

This ends the action of our construction during stage $e + 1$.

Recall that PT is built inductively by stages. At stage $e > 0$ we have produced a finite collection of p-strings S_e by extending p-strings from S_{e-1} and we restricted our tree PT at this stage to those p-strings compatible with p-strings from S_e. It is important to note that at any stage $e > 0$ each p-string from S_{e-1} is really extended to at least one (possibly more) p-string from S_e. This fact follows immediately from our construction. Thus, PT is an infinite perfect tree.

3.2. **Verification.** It remains to show that our construction achieves its aims. We must show that for any infinite path (X, Y) through PT which is computable from $\emptyset'$, $Y^{\geq 0} = X'$ and $A \leq_T X$. Since (X, Y) is recursive relative to $\emptyset'$ and X' is recursive in Y, X is low as required.

It is clear that during stage $e + 1$ we have decided the membership of e in X' for any such (X, Y) and so $Y^{\geq 0} = X'$. More precisely, using $\emptyset'$ we can find $\rho = (\sigma, \alpha) \in S_{e+1}$, $\sigma \prec X$ and then $e \in X'$ if and only if $\alpha(|\rho|) = \alpha(|\alpha|) = 1$. It remains only to verify $A \leq_T X$. For that it is sufficient to show that along any infinite path through PT there are only finitely many stages where an injury occurs.

For this purpose, we build a partial function H on $PR^{<\omega}$ (i.e. on p-strings) recursively in A. In the definition of H, we A-recursively approximate the $\emptyset'$-construction of PT. At the beginning, all p-strings are associated with an infinite recursive set M_Λ (the original set used for coding into $\mathcal{PA}$). During building H we sometimes either stop some strategies or restart strategies for defining values of H by changing current values of infinite recursive sets associated with p-strings, more precisely, p-strings preceding (σ, α) may eventually either stop a strategy for defining $H(\sigma, \alpha)$ or restart a strategy for defining $H(\sigma, \alpha)$ (if such strategy was not already finished) by changing a current value of M associated with (σ, α).

We will now first describe an isolated strategy of H for a p-string (σ, α) and an infinite recursive set M and then we will indicate how to combine strategies together.

Consider a tree $Tr_{(\sigma, \alpha)}(A, M)$. If σ is ω-extendable on $Tr_{(\sigma, \alpha)}(A, M)$ then the strategy has no output and no effect on p-strings extending (σ, α). If σ is not ω-extendable on $Tr_{(\sigma, \alpha)}(A, M)$, then define $H(\sigma, \alpha) = d_\emptyset$, where $d_\emptyset$ denotes the maximal d such that σ is d-extendable on $Tr_{(\sigma, \alpha)}(A, M)$. Further, wait for a step t such that $G(|\sigma| + 1, t) > d_j$ for both $j = 0, 1$, where d_j denotes the maximal d for which $\sigma * j$ is d-extendable on $Tr_{(\sigma, \alpha)}(A, M)$ (recall that $d_\emptyset = 1 + max(d_0, d_1)$). If there is no such t, the strategy has no effect on p-strings extending (σ, α). If there is such t, take the least such and denote it by t_0. Take a finite collection Q of all

p-strings $(\tau, \alpha * (-1)^{(t_0-1)} * j)$, for $j = 0, 1$ and $\tau \succ \sigma$ with $|\tau| - |\sigma| = t_0$, and stop all strategies (which are still active) for p-strings extending (σ, α) either with length $< |\sigma| + t_0$, or not compatible with any p-string from Q, and finally restart strategies for any p-string η of length $\geq |\sigma| + t_0$ compatible with some p-string from Q but now with a new infinite recursive set M which is determined as follows. For any p-string $\xi = (\tau, \alpha * (-1)^{(t_0-1)} * j)$ from Q, as we did in our construction, effectively find an infinite recursive set M^+ such that if $\mathcal{B}_\xi \neq \emptyset$ then $Restr(\mathcal{B}_\xi, M^+) = 2^\omega$, and ξ together with all p-strings extending such ξ are restarted with this newly associated set M^+.

All strategies for defining values of H are combined together easily by a finite injury style where any p-string has a higher priority than exactly all p-strings extending it. As mentioned previously, at the beginning all p-strings are associated with M_Λ (the original set used for coding into $\mathcal{PA}$).

We omit further details.

By virtue of its definition, H is an A-partial recursive function. The only values of H that are relevant for our purposes are those on p-strings from tree PT.

It remains to verify that $A \leq_T X$ for any infinite path (X, Y) through PT. Let such path (X, Y) be given. It is clearly sufficient to show that there are only finitely many p-strings $\rho_i, \rho_i \prec (X, Y)$, at which an injury occurs. Suppose, for a contradiction, that $\{\rho_i\}_i$ is an infinite sequence of p-strings with increasing length at which an injury occurred and for which $\rho_i \prec (X, Y)$. It follows from the construction and our assumptions on F^* and G that $H(\rho_i)$ is defined and greater than $F^*(|\rho_i|)$ for all i. Since $F^*(|\rho|) \geq F(\rho)$ for any p-string ρ, it immediately yields a failure of F to eventually dominate all A-partial recursive functions. A contradiction.

To finish the construction we have to provide an infinite path (X, Y) through PT which is recursive in $\emptyset'$, and then let $A^* = X$. This is easy. Recall that PT is an infinite perfect tree which is recursive in $\emptyset'$. Thus, e.g. we may fix the left-most (in the standard ordering) infinite path through PT. This path is obviously recursive in $\emptyset'$.

We end with a remark. The above construction can be carried out uniformly in a $\emptyset'$-index of A. That is, there are recursive functions f, g with properties stated in Lemma 3.5.

$\square$

Remark.   (1) In the proof of Lemma 3.5 we have coded a set A into A^* not straightforwardly. We could do that, but instead, we have nested into a Π_1^0 class $\mathcal{PA}$ a $\Pi_1^{0,A}$ class $\mathcal{PA}(A)$ all members of which are T-above A. This way does not increase a complexity of the proof and is more general. It can be used to provide just one global construction of a T-upper bound mentioned in Theorem 3.2.

(2) It is easy to verify that for any infinite path (X, Y) through PT (not necessarily computable by $\emptyset'$) in the above proof $A \leq_T X$, $X' = Y^{\geq 0}$ and $X \in GL_1$.

Before giving a proof of Theorem 3.2 we illustrate the main idea (of a more difficult implication) on a simple example.

Claim 3.6. *Given a sequence $\{A_n\}_n$ of low sets uniformly recursively in $\emptyset'$ so that low indices of all finite joins of members of the sequence are also uniformly recursive*

in $\emptyset'$ then there is a low set A which is T-above all A_n. Moreover, we may require $\mathbf{a} >> \mathbf{a_n}$ *for all n, where* $\mathbf{a}, \mathbf{a_n}$ *are T-degrees of A, A_n respectively.*

Proof. We give a sketch of the main idea. Let $B_n = \bigoplus_{i=0}^{n-1} A_i$ for $n \geq 1$ and $B_0 = \emptyset$. We construct recursively in $\emptyset'$ a sequence of nonempty relativized Π_1^0 classes $\{\mathcal{B}_n\}_n$ such that $\mathcal{B}_n \supseteq \mathcal{B}_{n+1}$ for all n and the required low set A recursive in $\emptyset'$ will be the only set in the intersection of all these classes. Each class $\mathcal{B}_n$ is a nonempty Π_1^{0,B_n} class. We also construct recursively in $\emptyset'$ a sequence of infinite recursive sets $\{M_e\}_e$ such that for all e, $M_e \supseteq M_{e+1}$ and $Restr(\mathcal{B}_e, M_e) = \mathcal{PA}(B_e)$.

Let $\mathcal{B}_0$ be $\mathcal{PA}$ and $M_0 = \omega$. So, since $B_0 = \emptyset$, $Restr(\mathcal{B}_0, M_0) = \mathcal{PA}(B_0)$.

At step $e + 1$ we first use the relativized Low Basis Theorem of Jockusch and Soare [7] to force the e-th instance of the jump operator. This gives a subclass $\mathcal{B}_{e+1}^* \subseteq \mathcal{B}_e$. Obviously $Restr(\mathcal{B}_{e+1}^*, M_e) \subseteq \mathcal{PA}(B_e)$ and $Restr(\mathcal{B}_{e+1}^*, M_e)$ is also a nonempty Π_1^{0,B_e} class. Thus, by a modification of Lemma 2.6, there is an infinite recursive set M_{e+1}, a subset of M_e, such that $Restr(\mathcal{B}_{e+1}^*, M_{e+1}) = 2^\omega$. Let $\mathcal{B}_{e+1} = \mathcal{B}_{e+1}^* \cap \{f : Restr(f, M_{e+1}) \in \mathcal{PA}(B_{e+1})\}$. Then, $\mathcal{B}_{e+1}$ is a $\Pi_1^{0,B_{e+1}}$ class and $Restr(\mathcal{B}_{e+1}, M_{e+1}) = \mathcal{PA}(B_{e+1})$.

This ends step $e + 1$.

One verifies that there is just only one set A in the intersection of all these classes and that A satisfies all required requirements. Moreover, T-degree of A is $>>$ T-degrees of all A_n. $\square$

Remark. We could equally construct, recursively in $\emptyset'$, an infinite perfect tree ($\subseteq$ $2^{<\omega}$) such that any infinite path through it both is T-above all A_n and belongs to GL_1. We could also work with elements of $2^\omega \times 2^\omega$ and build, recursively in $\emptyset'$, an infinite perfect tree ($\subseteq (\{0,1\} \times \{0,1\})^{<\omega}$) such that for any infinite path (X, Y) through it Y codes (in fact, is equal to) the jump of X.

However, in case we do not have low indices of such low sets A_n available (recursively in $\emptyset'$) and we have only a weaker information about lowness of these sets, we have to adapt the method explained in the proof of Claim 3.6 by combining it with the technique of the proof of Lemma 3.5. We explain subsequently how to do it.

Lemma 3.7. *Given a low set B with a low index b and a function F recursive in $\emptyset'$, there is a uniform way to obtain from a $\emptyset'$-index of a set A with the property that any partial function recursive in $B \oplus A$ is eventually dominated by F both a low set A^* and an index of lowness of A^* such that $B \oplus A \leq_T A^*$. That is to say that there are recursive functions f, g such that if $\Phi_e(\emptyset')$ is total and equal to some set A so that any partial function recursive in $B \oplus A$ is eventually dominated by F then $\Phi_{f(e)}(\emptyset')$ is a low set, $g(e)$ is its lowness index and $B \oplus A \leq_T \Phi_{f(e)}(\emptyset')$.*

Proof. It is an easy relativization of Lemma 3.5, i.e. the construction in the proof of Lemma 3.5 is relativized to an oracle B and it begins now with the class $\mathcal{PA}(B)$ instead of the class $\mathcal{PA}$. Let us explicitly note that we substantially use a low index b of B so that we can always decide whether a B-recursive tree ($\subseteq 2^{<\omega}$) is finite or infinite. Thus, we can always keep ω-extendability of relevant strings in infinite B-recursive trees, similarly as we kept ω-extendability of relevant strings in infinite recursive trees in the original construction (recall that, in the proof of Lemma 3.5, $(\sigma, \alpha) \in PT$ implies ω-extendability of σ in $Tr_{(\sigma,\alpha)}$). $\square$

The situation changes (slightly) if instead of a low index of a low set B we have only a weaker information about lowness of B, namely, if we have a function recursive in $\emptyset'$ which eventually dominates all partial function recursive in B.

Lemma 3.8. *Given a function F recursive in $\emptyset'$ and a low set B there is a uniform way to obtain from a $\emptyset'$-index of a set A with the property that any partial function recursive in $B \oplus A$ is eventually dominated by F both a low set A^* and an index of lowness of A^* such that $B \oplus A \leq_T A^*$. That is to say that there are recursive functions f, g such that if $\Phi_e(\emptyset')$ is total and equal to some set A so that any partial function recursive in $B \oplus A$ is eventually dominated by F then $\Phi_{f(e)}(\emptyset')$ is a low set, $g(e)$ is its lowness index and $B \oplus A \leq_T \Phi_{f(e)}(\emptyset')$.*

Proof. It is a further generalization of the technique used in the proof of Lemma 3.5 and Lemma 3.7. Suppose that any partial function recursive in $B \oplus A$ is eventually dominated by F. We have to replace missing low index of B by a technique of Lemma 3.5, i.e. we use guesses computed by F (or F^*) as approximations to correct answers to $\Sigma_1^{0,B}$ questions. By a finite injury style, these approximations are eventually really correct answers to such questions from some point on. From such a point the construction resembles that one described in the proof of Lemma 3.7. It means, the construction then simulates the proof of Lemma 3.7 with a deeper nested class $\mathcal{PA}(A)$ into the current version of $\mathcal{PA}(B)$. Thus, to combine everything together, we nest relativized Π_1^0 classes in a cascade style (like in Claim 3.6), here with depth of nesting equal to 2. We first nest a $\Pi_1^{0,B}$ class $\mathcal{PA}(B)$ into $\mathcal{PA}$ and then we further nest a $\Pi^{0,A}$ class $\mathcal{PA}(A)$ into it (by applying Lemma 2.6 like in the proof of Claim 3.6). Whenever (at any step) we encounter an injury at any level of nesting (along any p-string), we continue at the lowest level at which and also at all levels above it all guesses (computed by F, or F^*) currently look correct, we leave all current versions of nested classes at lower levels and restart all lower levels with new versions of nested classes. The crucial thing is that at the highest level (level 0), where we work with nonrelativized Π_1^0 class $\mathcal{PA}$, we never have any injury because $\emptyset'$ can always correctly decide any Σ_1^0 question. Under given assumptions the construction reaches all required goals. It uses standard finite injury technique and we omit further details. $\qquad\square$

Proof. We now complete the proof of Theorem 3.2. As it was already mentioned, the implication from (2) to (1) is direct.

To verify the implication from (1) to (2), we may identify the ideal $\mathcal{C}$ with the ideal $\mathcal{A}$ described in (1). Given a function F recursive in $\emptyset'$ and a sequence of sets $\{A_n\}_n$ which satisfy given assumptions, we combine the method of the proof of Claim 3.6 with the method of the proof of Lemma 3.8, i.e. missing low indices are replaced by the function F (or F^*). At each step we have to nest a next relativized Π_1^0 class into a previous one, where nested classes are $\mathcal{PA}(A_n)$ (or $\mathcal{PA}(\bigoplus_{j=0}^{n-1} A_j)$). Thus, the depth of nesting of relativized Π_1^0 classes increases by 1 at each step. Analogously as before, at each level of nesting there are only finitely many injuries. We omit further details.

$\qquad\square$

4. A question

We have left the following questions open.

Question 4.1. Is there a natural set of conditions which characterize whether an ideal in the Turing degrees of the recursively enumerable sets has a upper bound which is low and recursively enumerable?

Exact pairs play an important role in the study of T-degree structures. By a result of Nerode and Shore [13] it follows that there is an exact pair for the class of K-trivials in Δ_2^0 T-degrees. On the other hand the following problem is open.

Question 4.2. Is there a low exact pair for the class of K-trivial sets, i.e. are there low sets A, B such that $\mathcal{K} = \{C : C \leq_T A \ \& \ C \leq_T B\}$?

Finally, since the K-trivial sets are closely linked with the 1-random reals, the following special situation is interesting.

Question 4.3. Is there a low 1-random set which is a T-upper bound for the class of K-trivial sets?

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