EFFECTIVE RANDOMNESS FOR CONTINUOUS MEASURES

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ABSTRACT. We investigate which infinite binary sequences (reals) are effectively random with respect to some continuous (i.e., non-atomic) probability measure. We prove that for every n, all but countably many reals are n-random for such a measure, where n indicates the arithmetical complexity of the Martin-Löf tests allowed. The proof is based on a Borel determinacy argument and presupposes the existence of infinitely many iterates of the power set of the natural numbers. In the second part of the paper we present a metamathematical analysis showing that this assumption is indeed necessary. More precisely, there exists a computable function G such that, for any n, the statement "All but countably many reals are $G(n)$-random with respect to a continuous probability measure" cannot be proved in ZFC^-_n. Here ZFC^-_n stands for Zermelo-Fraenkel set theory with the Axiom of Choice, where the Power Set Axiom is replaced by the existence of n-many iterates of the power set of the natural numbers. The proof of the latter fact rests on a very general obstruction to randomness, namely the presence of an internal definability structure.

1. INTRODUCTION

The goal of this paper is study under what circumstances an infinite binary sequence (real) is random with respect to some probability measure. We use the framework of Martin-Löf randomness to investigate this question. Given a measure μ, a Martin-Löf test is an effectively presented G_δ μ-nullset in which the measure of the open sets converges effectively to zero. As there are only countably many such tests, only measure-zero many reals can be covered by a Martin-Löf test for μ. The reals that cannot be covered are called Martin-Löf random for μ. Obviously, if a real X is an atom of a measure μ, then X is random for μ. If we rule out this trivial way of being random, the task becomes harder: Given a real X, does there exist a probability measure

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on the space of all infinite binary sequences such that X is not an atom of μ but X is μ-random?

In [41], we were able to show that if a real X is not computable, then such a measure exists. It is not hard to see that if a real X is computable, then the only way that X is random with respect to a measure μ is for it to be an atom of μ. Hence having non-trivial random content (with respect to any measure at all) in the sense of Martin-Löf is equivalent to being non-computable. Besides Martin-Löf randomness, various other notions of algorithmic randomness have been thoroughly investigated, such as Schnorr randomness or Kurtz randomness. Two recent books on algorithmic randomness [9, 36] provide a good overview over the various concepts. They all have in common that they use algorithmic features to separate non-randomness from randomness. Moreover, in terms of the arithmetic hierarchy, the complexities of the underlying test notions usually fall within two or three quantifiers of each other.

This suggests that in order to study the random content of a real from the point of view of algorithmic randomness in general, we should look at how this content behaves when making tests more powerful by giving them access to oracles (or equivalently, considering nullsets whose definitions are more complicated). For Martin-Löf tests, this means the test has to be effectively G_δ only in some parameter Z. This enlarges the family of admissible nullsets and, correspondingly, shrinks the set of random reals. If the parameter Z is an instance $\emptyset^{(n)}$ of the Turing jump, i.e., Z is real that can decide all Σ_n statements about arithmetic, we speak of $(n+1)$-randomness.

Our goal is to understand the nature of the set of reals that are not n-random with respect to *any continuous probability measure*. In particular, we want to understand how this set behaves as n grows larger (and more reals will have this property).

The restriction to continuous measures makes sense for the following reasons. By a result of Haken [17, Theorem 5], if a real is n-random, $n > 2$, with respect to some (not necessarily continuous) probability measure and not an atom of the measure, it is $(n-2)$-random with respect to a continuous probability measure. Thus considering arbitrary probability measures would only shift the question of how random a real is by a couple of quantifiers. And the core problem of finding a measure that makes a real random without making the real an atom of the measure remains. While we ignore features of randomness for arbitrary measures at lower levels, we develop insights into randomness for continuous measures. At the level of 1-randomness, there is an interesting connection with computability theory: In [41], drawing on a result of Woodin [47], we showed that if a real X is not hyperarithmetic, then there exists a continuous probability measure for which X is 1-random.

Our first main result concerns the size of the set of reals that are not n-random with respect to any continuous measure. The case $n = 1$ follows of course from the result in [41] mentioned above.

Theorem 1. *For any $n \in \omega$, all but countably many reals are n-random with respect to some continuous probability measure.*

The proof features a metamathematical argument. Let us denote by NCR_n the set of all reals that are not n-random with respect to any continuous probability measure. We show that for each n, NCR_n is contained in a countable model of a fragment of set theory. More precisely, this fragment is ZFC^-_n, where ZFC^-_n denotes the axioms of Zermelo-Fraenkel set theory with the Axiom of Choice, with the power set axiom replaced by a sentence that assures the existence of n iterates of the power set of the natural numbers.

One may wonder whether this metamathematical argument is really necessary to prove the countability of a set of reals, in particular, whether one needs the existence of infinitely many iterates of the power set of ω to prove Theorem 1, a result about sets of reals. It turns out that this is indeed the case. This is the subject of our second main result.

Theorem 2. *There exists a computable function $G(n)$ such that for every $n \in \omega$, the statement*

> *"There exist only countably many reals that are not $G(n)$-random with respect to some continuous probability measure."*

is not provable in ZFC^-_n.

This metamathematical property of NCR is reminiscent of Borel determinacy [32]. Even before Martin proved that every Borel game is determined, Friedman [11] had shown that any proof of Borel determinacy had to use uncountably many iterates of the power set of ω. Borel determinacy is a main ingredient in our proof of Theorem 1. Theorem 2 establishes that this use is, in a certain sense, inevitable.

Theorem 2 is proved via a fine structure analysis of the countable models used to show NCR_n is countable. These models are certain levels L_β of Gödel's constructible hierarchy. In these L_β (or rather Jensen's version, the

J-hierarchy) we exhibit sequences of non-random reals with Turing degrees cofinal among those of the model. These reals are *master codes* [3, 22], reals that code initial segments of the J-hierarchy in a way that arithmetically reflects the strong stratification of L. The main feature of this proof is a very general principle that manifests itself in various forms: an internal stratified definability structure forms a strong obstruction to randomness. This principle works for both iterated Turing jumps as well as certain levels of the J-hierarchy.

Before we proceed, we make one more comment on the restriction to continuous measures. Note that Theorem 1 is a stronger statement for continuous measures than for arbitrary measures. Furthermore, by Haken's result [17], Theorem 2 holds for arbitrary measures if we replace $G(n)$ by $G(n) + 2$.

The paper is organized as follows. In Section 2, we introduce effective randomness for arbitrary (continuous) probability measures. We also prove some fundamental facts on randomness. In particular, we will give various ways to obtain reals that are random for *some* continuous measure from standard Martin-Löf random reals (i.e., random with respect to Lebesgue measure). We also consider the definability strength of random reals. Section 3 features the proof that for any n, all but countably many reals are n-random with respect to some continuous measure (Theorem 1). Finally, Section 4 is devoted to the metamathematical analysis of Theorem 1. In particular, it contains a proof of Theorem 2.

We expect the reader to have basic knowledge in mathematical logic and computability theory, including some familiarity with forcing, the constructible universe, and the recursion theoretic hierarchies.

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2. Randomness for Continuous Measures

In this section we review effective randomness on Cantor space 2^ω for arbitrary probability measures. We then prove some preliminary facts about random reals.

The *Cantor space* 2^ω is the set of all infinite binary sequences, also called *reals*. The topology generated by the *cylinder sets*

$$[\![\sigma]\!] = \{x : x\lceil_{|\sigma|} = \sigma\},$$

where σ is a finite binary sequence, turns 2^ω into a compact Polish space. $2^{<\omega}$ denotes the set of all finite binary sequences. If $\sigma, \tau \in 2^{<\omega}$, we use $\subseteq$ to denote the usual prefix partial ordering. This extends in a natural way to $2^{<\omega} \cup 2^\omega$. Thus, $x \in [\![\sigma]\!]$ if and only if $\sigma \subset x$. Finally, given $U \subseteq 2^{<\omega}$, we write $[\![U]\!]$ to denote the open set induced by U, i.e. $[\![U]\!] = \bigcup_{\sigma \in U} [\![\sigma]\!]$.

2.1. **Turing functionals.** While the concept of a Turing functional is standard, we will later define a forcing partial order based on it, and for this purpose we give a rather complete formal definition here. The definition follows [46], with the one difference that we require Turing functionals to be recursively enumerable.

A *Turing functional* Φ is a computably enumerable set of triples (m, k, σ) such that m is a natural number, k is either 0 or 1, and σ is a finite binary sequence. Further, for all m, for all k_1 and k_2, and for all compatible σ_1 and σ_2, if $(m, k_1, \sigma_1) \in \Phi$ and $(m, k_2, \sigma_2) \in \Phi$, then $k_1 = k_2$ and $\sigma_1 = \sigma_2$.

We will refer to a triple (m, k, σ) as a *computation* in Φ, and we will say it is a *computation along* X when every X is an extension of σ.

In the following, we will also assume that Turing functionals Φ are *use-monotone*, which means the following hold.

(1) For all (m_1, k_1, σ_1) and (m_2, k_2, σ_2) in Φ, if σ_1 is a proper initial segment of σ_2, then m_1 is less than m_2.
(2) For all m_1 and m_2, k_2 and σ_2, if $m_2 > m_1$ and $(m_2, k_2, \sigma_2) \in \Phi$, then there are k_1 and σ_1 such that $\sigma_1 \subseteq \sigma_2$ and $(m_1, k_1, \sigma_1) \in \Phi$.

We write $\Phi^\sigma(m) = k$ to indicate that there is a τ such that τ is an initial segment of σ, possibly equal to σ, and $(m, k, \tau) \in \Phi$. In this case, we also write $\Phi^\sigma(m)\downarrow$, as opposed to $\Phi^\sigma(m)\uparrow$, indicating that for all k and all $\tau \subseteq \sigma$, $(m, k, \tau) \notin \Phi$. If, moreover, (m, k, τ) is enumerated into Φ by time s, we write $\Phi^\sigma_s(m) = k$.

If $X \in 2^\omega$, we write $\Phi^X(m) = k$ (and $\Phi^X_s(m) = k$, respectively) to indicate that there is an l such that $\Phi^{X\lceil_l}(m) = k$ (and this is enumerated by time s, respectively). This way, for given $X \in 2^\omega$, Φ^X defines a partial function from

ω to $\{0,1\}$ (identifying reals with sets of natural numbers). If this function is total, it defines a real Y, and in this case we write $\Phi(X) = Y$ and say that Y is Turing reducible to X via Φ, $Y \leq_{\mathrm{T}} X$.

By use-monotonicity, if $\Phi^\sigma(m)\downarrow$, then $\Phi^\sigma(n)\downarrow$ for all $n < m$. If we let $\overline{m}$ be maximal such that $\Phi^\sigma(\overline{m})\downarrow$, Φ^σ gives rise to a string τ of length $\overline{m}+1$,

$$\tau = \Phi^\sigma(0)\dots\Phi^\sigma(\overline{m}).$$

If $\Phi^\sigma(n)\uparrow$ for all n, we put $\tau = \varnothing$. On the other hand, if $\overline{m}$ does not exist, then Φ^σ gives rise to a real Y. We write $\Phi(\sigma) = \tau$ or $\Phi(\sigma) = Y$, respectively. This way a Turing functional induces a function from $2^{<\omega}$ to $2^{<\omega} \cup 2^\omega$ that is *monotone*, that is, $\sigma \subseteq \tau$ implies $\Phi(\sigma) \subseteq \Phi(\tau)$. Note that $\Phi(\sigma)$ is not necessarily a computable function, but we can effectively approximate it by prefixes. More precisely, there exists a computable mapping $(\sigma, s) \mapsto \Phi_s(\sigma)$ so that $\Phi_s(\sigma) \subseteq \Phi_{s+1}(\sigma)$, $\Phi_s(\sigma) \subseteq \Phi_s(\sigma^\frown i)$ for $i \in \{0,1\}$, and $\lim_s \Phi_s(\sigma) = \Phi(\sigma)$.

If, for a real X, $\lim_n |\Phi(X\lceil_n)| = \infty$, then $\Phi(X) = Y$, where Y is the unique real that extends all $\Phi(X\lceil_n)$. In this way, Φ also induces a partial, continuous function from 2^ω to 2^ω. We will use the same symbol Φ for the Turing functional, the monotone function from $2^{<\omega}$ to $2^{<\omega}$, and the partial, continuous function from 2^ω to 2^ω. It will be clear from the context which Φ is meant. Φ is called *total* if $\Phi(X)$ is a real for all $X \in 2^\omega$. If Φ is total and $\Phi(X) = Y$, then Y is called *truth-table reducible* to X, $Y \leq_{\mathrm{tt}} X$.

Turing functionals can be relativized with respect to a parameter Z, by requiring that Φ is r.e. in Z. We call such functionals *Turing Z-functionals*. This way we can consider relativized Turing reductions. A real X is Turing reducible to a real Y relative to a real Z, written $X \leq_{\mathrm{T}(Z)} Y$, if there exists a Turing Z-functional Φ such that $\Phi(X) = Y$.

2.2. **Probability measures.** By the Carathéodory extension theorem, a Borel probability measure μ on 2^ω is completely specified by its values on *clopen sets*, i.e., on finite unions of basic open cylinders. In particular, $\mu[\![\varnothing]\!]) = 1$, and the additivity of μ implies that for all $\sigma \in 2^{<\omega}$,

$$\mu[\![\sigma]\!] = \mu[\![\sigma^\frown 0]\!] + \mu[\![\sigma^\frown 1]\!]. \tag{2.1}$$

An *additive premeasure* is a function $\eta : 2^{<\omega} \to \mathbb{R}^{\geq 0}$ with $\eta(\varnothing) = 1$ and $\eta(\sigma) = \eta(\sigma^\frown 0) + \eta(\sigma^\frown 1)$ for all $\sigma \in 2^{<\omega}$. Any additive premeasure induces a Borel probability measure, and if we restrict a Borel probability measure μ to its values on cylinders, we obtain an additive premeasure whose Carathéodory extension is μ. We can therefore identify a Borel probability measure on 2^ω with the additive premeasure it induces. We will exclusively deal with Borel probability measures and in the following simply write *measure* to denote a Borel probability measure on 2^ω.

The *Lebesgue measure* λ on 2^ω is obtained by distributing a unit mass uniformly along the paths of 2^ω, i.e., by setting $\lambda[\![\sigma]\!] = 2^{-|\sigma|}$. A *Dirac measure*, on the other hand, is defined by putting a unit mass on a single real, i.e., for $X \in 2^\omega$, let

$$\delta_X[\![\sigma]\!] = \begin{cases} 1 & \text{if } \sigma \subset X, \\ 0 & \text{otherwise.} \end{cases}$$

If, for a measure μ and $X \in 2^\omega$, $\mu(\{X\}) > 0$, then X is called an *atom* of μ. Obviously, X is an atom of δ_X. A measure that does not have any atoms is called *continuous*.

2.3. **Representation of measures and Martin-Löf randomness.** To incorporate measures into an effective test for randomness we represent them as reals. This can be done in various ways (for example, identify them with the underlying premeasure and code that), but in order for the main arguments in [41] to work, the representation has to reflect some of the topological properties of the space of probability measures.

Let $\mathcal{M}(2^\omega)$ be the set of all Borel probability measures on 2^ω. With the weak-* topology, this becomes a compact Polish space (see [26, Theorem 17.23]). It is possible to choose a countable dense subset $\mathcal{D} \subseteq \mathcal{M}(2^\omega)$ so that every measure in $\mathcal{M}(2^\omega)$ is the limit of an effectively converging Cauchy sequence of measures in $\mathcal{D}$. Moreover, the structure of the measures in $\mathcal{D}$ is such that they give rise to a canonical continuous surjection $\rho : 2^\omega \to \mathcal{M}(2^\omega)$ with the additional property that for every $R \in 2^\omega$, $\rho^{-1}(\{\rho(R)\})$ is $\Pi^0_1(R)$. For details on the construction of ρ, see [4, Section 2]. If $\rho(R) = \mu$, then R is called a *representation* of μ. A measure may have several distinct representations with respect to ρ. If μ is given, R_μ will always denote a representation of μ.

Working with representations, we can apply computability theoretic notions to measures. The following two observations appeared as Propositions 2.2 and 2.3, respectively, in [41].

Proposition 2.1. *Let $R \in 2^\omega$ be a representation of a measure $\mu \in \mathcal{M}(2^\omega)$. Then the relations*

$$\mu[\![\sigma]\!] < q \quad \text{and} \quad \mu[\![\sigma]\!] > q \qquad (\sigma \in 2^{<\omega}, q \in \mathbb{Q})$$

are r.e. in R.

It follows that the representation of a measure can effectively approximate effectively approximate its values on cylinders to arbitrary precision.

Proposition 2.2. *Let $R \in 2^\omega$ be a representation of a measure $\mu \in \mathcal{M}(2^\omega)$. Then R computes a function $g_\mu : 2^{<\omega} \times \omega \to \mathbb{Q}$ such that for all $\sigma \in 2^{<\omega}$, $n \in \omega$,*

$$|g_\mu(\sigma, n) - \mu[\![\sigma]\!]| \leq 2^{-n}.$$

We say a real X is *recursive in* μ if $X \leq_{\mathrm{T}} R_\mu$ for every representation R_μ of μ. On the other hand, we say a real computes a measure if its computes *some* representation of it. A measure does not necessarily have a representation of least Turing degree [4, Theorem 4.2].

We will later show that the question of whether a real is random with respect to a continuous measure can be reduced to considering only continuous dyadic measures. A measure μ is *dyadic* if every measure of a cylinder is of the form $\mu[\![\sigma]\!] = m/2^n$ with m, n non-negative integers.

For dyadic measures, it makes sense to speak of *exact computability*: A dyadic measure μ is *exactly computable* if the function $\sigma \mapsto \mu[\![\sigma]\!]$ is a computable mapping from $2^{<\omega}$ to $\mathbb{Q}$. Note that for exactly computable measures, the relation $\mu[\![\sigma]\!] > \alpha$, α rational, is decidable, whereas in the general case for μ with a computable representation we only know it is Σ^0_1.

If we encode a dyadic measure μ by collecting the ternary expansions of its values on cylinders in a single real Y, we obtain a representation not in the sense of ρ, but that is minimal in the following sense: Any real that can compute an approximation function to μ in the sense of Proposition 2.2 can compute Y.

We can now give the definition of a general Martin-Löf test. The definition is a generalization of Martin-Löf n-tests and Martin-Löf n-randomness for Lebesgue measure. We relativize both with respect to a representation of the measure and an additional parameter.

Definition 2.3. Suppose μ is a probability measure on 2^ω, and R is a representation of μ. Suppose further that $Z \in 2^\omega$ and $n \geq 1$.

(1) An (R, Z, n)-*test* is a set $W \subseteq \omega \times 2^{<\omega}$ which is recursively enumerable in $(R \oplus Z)^{(n-1)}$, the $(n-1)$st Turing jump of $R \oplus Z$ such that

$$\sum_{\sigma \in W_n} \mu[\![\sigma]\!] \leq 2^{-n},$$

where $W_n = \{\sigma : (n, \sigma) \in W\}$

(2) A real X *passes* a test W if $X \notin \bigcap_n [\![W_n]\!]$. If X does not pass a test W, we also say X is *covered* by W (or (W_n), respectively).

(3) A real X is (R, Z, n)-*random* if it passes all (R, Z, n)-tests.

(4) A real X is *Martin-Löf n-random for μ relative to Z*, or simply (μ, Z, n)-*random* if there exists a representation R_μ such that X is (R_μ, Z, n)-random. In this case we say R_μ *witnesses* the μ-randomness of X.

If the underlying measure is Lebesgue measure λ, we often drop reference to the measure and simply say X is (Z, n)-*random*. We also drop the index 1 in case of $(\mu, Z, 1)$-randomness and simply speak of *μ-randomness relative to Z* or *μ-Z-randomness*. If $Z = \emptyset$, on the other hand, we speak of (μ, n)- or *μ-n-randomness*. Note also that if μ is Z-computable, say $R_\mu \leq_{\mathrm{T}} Z$, then (R_μ, Z, n)-randomness is the same as $(R_\mu, Z^{(n-1)}, 1)$-randomness.

Remark 2.4. The original definition of n-randomness for Lebesgue measure λ given by Kurtz [29] uses tests based on Σ_n classes. However, it is possible to approximate Σ^0_n classes from outside in measure by open sets. Kurtz [29] and Kautz [25] showed that such an approximation in measure can be done effectively for classes of the lightface finite Borel hierarchy, in the sense that a Σ^0_n class can be approximated in measure by a $\Sigma^{0,\emptyset^{(n-1)}}_1$ class. Therefore, while the definitions based on Σ^0_n nullsets and on $\Sigma^{0,\emptyset^{(n-1)}}_1$ nullsets do not give the same notion of test, they yield the same class of random reals (see [9, Section 6.8] for a complete presentation of this argument).

The proof that the two approaches yield the same notion of n-randomness relativizes. Moreover, the approximation in measure by open sets is possible for any Borel probability measure, as any finite Borel measure on a metric space is regular. Finally, using Proposition 2.1 one can show inductively that, given a representation R of μ, the relations $\mu(S) > q$ and $\mu(S) < q$ (for q rational) are uniformly $\Sigma^{0,R}_n$ for any Σ^0_n class S. The latter fact is a key ingredient in the equivalence proof. Therefore, (μ, Z, n)-randomness could alternatively be defined via $\Sigma^{0,R\oplus Z}_n$ tests. We prefer the approach given in

Definition 2.3, because open sets are usually easier to work with, and because most techniques relativize.

Levin [30] introduced the alternative concept of a *uniform test* for randomness, which is representation-independent (see also [13]). Day and Miller [4, Theorem 1.6] have shown that for any measure μ and for any real X, X is μ-random in the sense of Definition 2.3 if and only if X is μ-random for uniform tests.

Since, for fixed R_μ, Z, and n, there are only countably many (R_μ, Z, n)-tests, it follows from countable additivity that the set of (μ, Z, n)-random reals for any μ and any Z has μ-measure 1. Hence there always exist (μ, Z, n)-random reals for any measure μ, any real Z, and any $n \geq 1$.

However, μ, Z, and n put some immediate restrictions on the relative definability of any (μ, Z, n)-random real.

Proposition 2.5. *If X is (μ, Z, n)-random via a representation R_μ, then X cannot be $\Delta^0_n(R_\mu \oplus Z)$.*

Proof. If X is $\Delta^0_n(R_\mu \oplus Z)$, then $X \leq_{\mathrm{T}} (R_\mu \oplus Z)^{(n-1)}$, and we can build a (μ, Z, n)-test covering X by using the cylinders given by its initial segments. □

It is also immediate from the definition of randomness that any atom of a measure is random with respect to it. This is a trivial way for a real to be random. The proposition below (a straightforward relativization of a result by Levin [48], see also [41, Proposition 3.3]) shows that atoms of a measure are also computationally trivial (relative to the measure).

Proposition 2.6 (Levin)**.** *If for a measure μ and a real X, $\mu\{X\} > 0$, then $X \leq_{\mathrm{T}} R_\mu$ for any representation R_μ of μ.*

Since we are interested in randomness for continuous measures, the case of atomic randomness is excluded a priori.

2.4. **Image measures and transformation of randomness.** Let $f : 2^\omega \to 2^\omega$ be a Borel measurable function. If μ is a measure on 2^ω ,the *image measure* μ_f is defined by

$$\mu_f(A) = \mu(f^{-1}(A)).$$

It can be shown that every probability measure is the image measure of Lebesgue measure λ for some f. For continuous measures, Oxtoby [37] proved that any continuous, positive measure on 2^ω can be transformed into Lebesgue measure on the set of irrationals in $[0,1]$ via a homeomorphism. Here, a measure is *positive* if $\mu[\![\sigma]\!] > 0$ for $\sigma \in 2^{<\omega}$.

Levin [48, Theorem 4.3], and independently Kautz [25, Corollary IV.3.18] (see also [2]) proved an effective version of these results. For a computable measure μ on 2^ω there exists a Turing functional Φ defined on almost every real such that μ is the image measure of λ under Φ. If μ is, moreover, continuous and positive, then Φ has an inverse that transforms μ into λ.

A consequence of the Levin-Kautz theorem is that every non-recursive real that is random with respect to a computable probability measure is Turing equivalent to a λ-random real. We will show now that for continuous measures, this can be strengthened to *truth-table equivalence*.

Proposition 2.7. *Let X be a real. For any $Z \in 2^\omega$ and any $n \geq 1$, the following are equivalent.*

(i) X is (μ, Z, n)-random for a continuous measure μ recursive in Z.
(ii) X is (ν, Z, n)-random for a continuous, positive, dyadic measure ν exactly computable in Z.
(iii) There exists a Turing Z-functional Φ such that Φ is an order-preserving homeomorphism of 2^ω, and $\Phi(X)$ is (λ, Z, n)-random.
(iv) X is truth-table equivalent relative to Z to a (λ, Z, n)-random real.

Here, the order on 2^ω is the *lexicographical order* given by

$$X < Y \quad :\Leftrightarrow \quad X(N) < Y(N) \text{ where } N = \min\{n \colon X(n) \neq Y(n)\}.$$

Proof. We give a proof for $Z = \emptyset$ and $n = 1$. It is routine to check that the proof relativizes and generalizes to higher levels of randomness.

(i) $\Rightarrow$ (ii): Let X be μ-random, where μ is a continuous, computable measure. We construct a continuous, positive, dyadic, and exactly computable measure ν such that X is random with respect to ν, too. The construction is similar to Schnorr's rationalization of martingales [44] (see also [9, Proposition 7.1.2]).

We define ν by recursion on the full binary tree $2^{<\omega}$. To initialize, let $\nu^*[\![\varnothing]\!] = 2$. Now assume $\nu^*[\![\sigma]\!]$ is defined such that,

$$\mu[\![\sigma]\!] < \nu^*[\![\sigma]\!] < \mu[\![\sigma]\!] + 2^{-|\sigma|+1}.$$

A simple case distinction shows that

$$\max\{\mu[\![\sigma^\frown 0]\!], \nu^*[\![\sigma]\!] - \mu[\![\sigma^\frown 1]\!] - 2^{-|\sigma|}\} < \min\{\mu[\![\sigma^\frown 0]\!] + 2^{-|\sigma|}, \nu^*[\![\sigma]\!] - \mu[\![\sigma^\frown 1]\!]\}.$$

As the dyadic rationals are dense in $\mathbb{R}$, there exists a dyadic rational r in this interval, and by Proposition 2.2 we can find such an r effectively in σ. Put

$$\nu^*[\![\sigma^\frown 0]\!] = r, \quad \nu^*[\![\sigma^\frown 1]\!] = \nu^*[\![\sigma]\!] - r.$$

Then clearly

$$\nu^*[\![\sigma^\frown 0]\!] + \nu^*[\![\sigma^\frown 1]\!] = \nu^*[\![\sigma]\!],$$

and by the choice of r,

$$\mu[\![\sigma^\frown 0]\!] < \nu^*[\![\sigma^\frown 0]\!] < \mu[\![\sigma^\frown 0]\!] + 2^{-(|\sigma|)},$$
$$\mu[\![\sigma^\frown 1]\!] < \nu^*[\![\sigma^\frown 1]\!] < \mu[\![\sigma^\frown 1]\!] + 2^{-(|\sigma|)}.$$

We normalize by letting $\nu = \nu^*/2$. By construction of ν^*, the measure ν is dyadic and exactly computable. It is also clear from the construction that for all σ, $\mu[\![\sigma]\!] < 2\nu[\![\sigma]\!]$. In particular, ν is positive. Finally, if (V_n) is a test for ν, by letting $W_n = V_{n+1}$ we obtain a μ-test that covers every real covered by (V_n). Hence if X is μ-random, then X is also ν-random.

(ii) $\Rightarrow$ (iii): Suppose ν is an exactly computable, continuous, positive, dyadic measure. Since ν is continuous and 2^ω is compact, for every m there exists a least $l_m \in \omega$ such that whenever $|\sigma| \geq l_m$, then $\nu[\![\sigma]\!] \leq 2^{-m}$. Without loss of generality, we can assume that $l_m < l_{m+1}$. As ν is exactly computable, the mapping $m \mapsto l_m$ is computable.

We define inductively a mapping $\varphi : 2^{<\omega} \to 2^{<\omega}$ that will induce the desired homeomorphism. In order to do so, we first define, for every $\tau \in 2^{<\omega}$, an auxiliary finite, non-empty set $E_\tau \subseteq 2^{<\omega}$. It will hold that

(a) all strings in E_τ are of the same length, and this length depends only on the length of τ;
(b) if $\sigma \subseteq \tau$, then every string in E_τ is an extension of some string in E_σ;
(c) if σ and τ are incomparable, then E_σ and E_τ are disjoint; moreover, if $|\sigma| = |\tau|$ and σ is lexicographically less than τ, then all strings in E_σ are lexicographically less than any string in $E\tau$;
(d) for all n, $\bigcup_{|\tau|=n}[\![E_\tau]\!] = 2^\omega$;
(e) for all τ, $0 < \nu[\![E_\tau]\!] \leq 2^{-|\tau|}(2 - 2^{-|\tau|})$.

Put $E_\varnothing = \{\varnothing\}$. Suppose now that E_τ is defined for all strings τ of length at most n, and that for these sets $E\tau$, (a)-(e) are satisfied.

Given any τ of length n, let

$$F_\tau = \{\sigma : |\sigma| = l_{2(n+1)} \;\&\; \sigma \text{ extends some string in } E_\tau\}.$$

Find the least (with respect to the usual lexicographic ordering) $\sigma \in F_\tau$ such that

$$\sum_{\substack{\eta \leq \sigma \\ \eta \in F_\tau}} \nu[\![\eta]\!] \geq \nu[\![E_\tau]\!]/2.$$

Let $E_{\tau^\frown 0} = \{\eta \in F_\tau : \eta < \sigma\}$ and put the remaining strings of F_τ into $E_{\tau^\frown 1}$. This ensures that $E_{\tau^\frown 0}$ and $E_{\tau^\frown 1}$ satisfy (a), (b), (c), and (d). Moreover, by the choice of the length of strings in F_σ and property (e) for E_τ, both $E_{\tau^\frown 0}$ and $E_{\tau^\frown 1}$ are non-empty. As ν is positive, this implies $\nu[\![E_{\tau^\frown i}]\!] > 0$ for each $i \in \{0, 1\}$. Moreover, for each $i \in \{0, 1\}$, we can use the induction hypothesis for $E\sigma$ and deduce that

$$\begin{aligned} \nu[\![E_{\tau^\frown i}]\!] &\leq \nu[\![E_\tau]\!]/2 + 2^{-2|\tau|-1} \\ &\leq 2^{-|\tau|-1}(2 - 2^{-|\tau|}) + 2^{2(|\tau|+1)} \\ &= 2^{-|\tau|-1}(2 - 2^{-|\tau|-1}), \end{aligned}$$

which yields the bound in (d).

Now we define the mapping φ: Put $\varphi(\varnothing) = \varnothing$. Suppose now $\varphi(\sigma)$ is defined for all τ of length less than or equal to n. Given τ of length n, map all strings in $E_{\tau^\frown 0}$ to $\tau^\frown 0$, and all strings in $E_{\tau^\frown 1}$ to $\tau^\frown 1$. To make φ defined on all strings, map any string that extends some string in E_τ but is a true prefix of some string in F_τ to τ.

It is clear from the construction that φ induces a total, order preserving mapping $\Phi : 2^\omega \to 2^\omega$ by letting

$$\Phi(X) = \lim_n \varphi(X\lceil_n).$$

Φ is onto since for every σ, E_σ is not empty. We claim that Φ is also one-one. Suppose $\Phi(X) = \Phi(Y)$. This implies that for all n, $\varphi(X\lceil_n) = \varphi(Y\lceil_n)$, that is, for all n, $X\lceil_n$ and $Y\lceil_n$ belong to the same E_σ. Since ν is positive, the diameter of the E_σ goes to 0 along any path. Hence $X = Y$.

It remains to show that $\Phi(X)$ is Martin-Löf random. Suppose not, then there exists an λ-test (W_n) that covers $\Phi(X)$. Let

$$V_n = \bigcup_{\sigma \in W_{n+2}} E_\sigma.$$

Then (V_n) covers X. Furthermore, the (V_n) are uniformly enumerable since the mapping $\sigma \mapsto E_\sigma$ is computable by the construction of the E_σ. Finally,

$$\sum_{\tau \in V_n} \nu[\![\tau]\!] = \sum_{\sigma \in W_{n+2}} \nu[\![E_\sigma]\!] \leq \sum_{\sigma \in W_{n+2}} 2^{-|\sigma|+2} \leq 2^{-n}.$$

thus X is not ν-random, contradiction.

(iii) $\Rightarrow$ (iv): This is immediate.

(iv) $\Rightarrow$ (i): This follows from Theorem 5.7 in [41]

□

The result also suggests that if we are only interested in whether a real is random with respect to a continuous measure, representational issues do not really arise. We can restrict ourselves to dyadic measures, which have a minimal representation.

Remark 2.8. We will henceforth, unless explicitly noted, assume that *any measure is a dyadic measure*. We drop reference to the representation and write μ instead of R_μ.

2.5. **Continuous randomness via Turing reductions.** While Proposition 2.7 gives a necessary and sufficient criterion for reals being random for a continuous measure, we will later need further techniques to show that a given real is random with respect to a continuous measure. As many of our arguments will involve arithmetic definability, it will be helpful to know to what extend randomness for continuous measures can be "transfered" via Turing reductions instead of truth-table reductions. The key ingredients are a theorem by Demuth [6] and a result by Kurtz [29].

Demuth [6, Theorem 17] showed that every non-recursive real truth-table below a Martin-Löf random real measure is Turing equivalent to a Martin-Löf-random real. The proof relativizes (as can be seen from the presentation in [9, Theorems 6.12.9 and 8.6.1]) and yields the next proposition.

Recall that we only consider dyadic measures and hence drop reference to a representation. Nevertheless, the results in this section are not dependent on the existence of a minimal representation and can be reformulated accordingly.

Proposition 2.9 (Demuth)**.** *Suppose Y is (μ, Z, n)-random ($n \geq 1$) and X is truth-table reducible to Y relative to $(\mu \oplus Z)^{(k)}$ for some $k \leq n-1$ (i.e., $X \leq_{\mathrm{tt}((\mu\oplus Z)^{(k)})} Y$). Further suppose X is not recursive in $(\mu \oplus Z)^{(k)}$. Then X is Turing equivalent relative to $(\mu \oplus Z)^{(k)}$ to a $(\lambda, \mu \oplus Z, n)$-random real.*

Kurtz [29, Theorem 4.3] observed that 2-random reals are $\emptyset'$-dominated. More precisely, there exists a $\emptyset'$-computable function dominating every function computable from a 2-random real.

The proof is based on the following idea (see [36, Proposition 5.6.28]): Given a Turing functional Φ, $\emptyset'$ can decide, given rational q and $n \in \omega$, whether

$$\lambda\{Y : \Phi(Y)(k) \text{ is defined for all } k \leq n\} > q.$$

For each n, let q_n be maximal of the form $i \cdot 2^{-n}$ so that the above holds, and let t_n be such that Φ converges on at least measure q_n-many strings of length t_n by time t_n. Construct a function $f \leq_{\mathrm{T}} \emptyset'$ such that $f(n)$ dominates all function values $\Phi(Y)$ computed with use t_n and within t_n steps. Then the set of all Y for which $\Phi(Y)$ is total and not dominated by f has Lebesgue measure 0 and can be captured by a $\emptyset'$-Martin-Löf-test. The argument relativizes to other measures and parameters, and we obtain the following.

Proposition 2.10 (Kurtz)**.** *Given a measure μ and a real Z, there exists a function $f \leq_{\mathrm{T}} (\mu \oplus Z)'$ such that for every $(\mu, Z, 2)$-random X, if $g \leq_{\mathrm{T}(Z \oplus \mu)} X$, then g is dominated by f.*

Together with Proposition 2.9 this yields a sufficient criterion for continuous randomness.

Lemma 2.11. *Suppose $n \geq 3$ and Y is (μ, Z, n)-random. If $X \leq_{\mathrm{T}(\mu \oplus Z)} Y$ and X is not recursive in $(\mu \oplus Z)'$, then X is $(\nu, (\mu \oplus Z)'', n-2)$-random for some continuous measure $\nu \leq_{\mathrm{T}} (\mu \oplus Z)''$.*

Proof. We assume $Z = \emptyset$ to keep notation simple. Suppose $X \leq_{\mathrm{T}(\mu)} Y$ via a Turing reduction Φ. By Proposition 2.10, the use and the convergence time of Φ on Y are dominated by some function recursive in μ'. We can modify Φ to $\widetilde{\Phi}$ such that $\widetilde{\Phi}$ is a truth-table reduction relative to μ' and $\widetilde{\Phi}(Y) = X$.

By Proposition 2.9, X is Turing equivalent relative to μ' to a (λ, μ, n)-random real L. Any (λ, μ, n)-random real is also $(\lambda, \mu', n-1)$-random, and so we can apply Proposition 2.10 to X and L to conclude that they are truth-table equivalent relative to μ''. This in turn means that X is truth-table equivalent relative to μ'' to a $(\lambda, \mu'', n-2)$-random real, which by Proposition 2.7 implies that X is $(\nu, \mu'', n-2)$-random for a continuous measure recursive in μ''. □

2.6. **The definability strength of randomness.** Lemma 2.11 shows that sufficiently high randomness for continuous measures propagates downward under Turing reductions (losing some of the randomness strength, however). This result was partly based on the fact that, for $n \geq 2$, n-random reals cannot compute fast-growing functions (beyond what is computable by $\emptyset'$). There is further evidence that the computational strength of n-random reals is rather limited.

For example, random reals are generalized low (relative to the measure). This is a generalization of a result due to Kautz [25, Theorem III.2.1].

Proposition 2.12 (Kautz)**.** *Let μ be a continuous measure, and suppose X is μ-$(n+1)$-random, where $n \geq 1$. Then*

$$(X \oplus \mu)^{(n)} \equiv_{\mathrm{T}} X \oplus \mu^{(n)}$$

The generalization works for the same reasons that n-randomness can be defined equivalently in terms of $\Sigma_n^{0,\mu}$-tests or $\Sigma_1^{0,\mu^{(n-1)}}$-tests: Borel probability measures are regular, and the relations $\mu(S) > q$ and $\mu(S) < q$ (for q rational) are uniformly $\Sigma_n^{0,\mu}$ for any Σ_n^0 class S.

Furthermore, one can generalize a result of Downey, Nies, Weber, and Yu [8], who show that every weakly 2-random real forms a minimal pair with $0'$. This will be of central importance in Section 4. For our purposes, it suffices to consider randomness instead of weak randomness, which we do in the following lemma.

Lemma 2.13. *Suppose μ is a continuous measure and Y is μ-n-random, $n \geq 2$. If $X \leq_{\mathrm{T}} \mu^{(n-1)}$ and $X \leq_{\mathrm{T}} Y \oplus \mu$, then $X \leq_{\mathrm{T}} \mu$.*

The structure of the proof is as follows: Following Downey, Nies, Weber, and Yu, we first show that the upper cone by Φ is Π_2^0 (relative to $\mu^{(n-2)}$). Next, we argue that the upper cone has cannot have measure zero since it contains a random real. Finally, one uses this fact to isolate X as a path in a μ-r.e. tree. The last step is a generalized version of the result that if the Turing upper cone of a real has positive Lebesgue measure, then the real must be computable [5, 42]. Our presentation follows [36].

Proof of Lemma 2.13. Suppose $X \leq_{\mathrm{T}} Y \oplus \mu$ via a Turing functional Φ and $Y \leq_{\mathrm{T}} \mu^{(n-1)}$. Note that Y is Δ_2^0 relative to $\mu^{(n-2)}$. Let $Y(n,s)$ be a $\mu^{(n-2)}$-recursive approximation of Y, i.e., $\lim_s Y(n,s) = Y(n)$. Given $i,s \in \omega$, put

$$U_{i,s} = \{X \colon \exists t > s \left(\Phi_t^{X \oplus \mu}(i) = Y(i,t)\right)\}.$$

The set $U_{i,s}$ is $\Sigma_1^{0,\mu^{(n-2)}}$ uniformly in i,s and hence $P = \bigcap_{i,s} U_{i,s}$ is $\Pi_2^{0,\mu^{(n-2)}}$. Note that P is the upper cone of X under Φ,

$$P = \{A \colon \Phi(A) = X\}.$$

P cannot have μ-measure 0: If it had then, since Borel probability measures on 2^ω are regular, for the sequence of open sets $(V_k)_{k\in\omega}$ given by $V_k = \bigcap_{\langle i,s\rangle \leq k} U_{i,s}$, we have $\mu V_k \searrow 0$. Since each V_k is $\Sigma_1^{0,\mu^{(n-2)}}$, $\mu^{(n-1)}$ can decide whether $\mu V_k \leq 2^{-l}$ for given l. Hence, we can convert (V_k) into a (μ, n)-test. Since $\bigcap_k V_k = P$ and P contains Y, this contradicts the fact that Y is μ-n-random.

Hence pick r rational such that $\mu P > r > 0$, where r is rational. Define a tree T by letting

$$\sigma \in T \;:\Leftrightarrow\; \mu\{\tau \colon \Phi(\tau \oplus \mu) \supseteq \sigma\} > r,$$

and closing under initial segments. T is r.e. in μ and X is an infinite path through T.

Since μ is a probability measure, any antichain in T contains at most $\lceil 1/r \rceil$ strings. Choose $\sigma = X\lceil_n$ such that no $\tau \supseteq \sigma$ incompatible with X is in T. Such σ exists for otherwise we could find an antichain of more than $\lceil 1/r \rceil$ strings branching off X. To compute $X\lceil_m$ from μ, it suffices to enumerate T above σ until a long enough extension shows up. □

We will later need the following relativization of the previous lemma. The proof is similar.

Lemma 2.14. *Suppose μ is a continuous measure and Z is μ-$(k+n)$-random, $k \geq 0, n \geq 2$. If $Y \leq_{\mathrm{T}} \mu^{(k+n-1)}$ and $Y \leq_{\mathrm{T}} Z \oplus \mu^{(k)}$, then $Y \leq_{\mathrm{T}} \mu^{(k)}$.*

One interpretation of Lemmas 2.13 and 2.14 is that μ-random reals are not helpful in computing (defining) reals arithmetic in μ. For example, if a real is properly Δ^0_n relative to a measure μ, then it cannot be Δ^0_k relative to $\mu \oplus X$ where $k < n$ and X is μ-$(n+1)$-random.

In Section 4, we will also need a result similar to the previous lemmas regarding initial segments of linear orders, namely, that random reals are not helpful in the recognizing well-founded initial segments. The following lemma may appear technical at this point, but its importance will become clear towards the end of Section 4, in the proof of Theorem 4.50.

Lemma 2.15. *Let $j \geq 0$. Suppose μ is a continuous measure and $\prec$ is a linear order on a subset of ω such that the relation $\prec$ and the field of $\prec$ are both recursive in $\mu^{(j)}$. Suppose further X is $(j+5)$-random relative to μ, and $I \subseteq \omega$ is the longest well-founded initial segment of $\prec$. If I is recursive in $(X \oplus \mu)^{(j)}$, then I is recursive in $\mu^{(j+4)}$.*

Proof. Suppose $I \leq_{\mathrm{T}} (X \oplus \mu)^{(j)}$, X is $(j+5)$-random relative to μ, but $I \not\leq_{\mathrm{T}} \mu^{(j+4)}$. By Lemma 2.11 (where $X, \mu^j, 5, I$ are substituted for Y, Z, n, X in the statement of the Lemma, respectively), there is a continuous measure $\mu_I \leq_{\mathrm{T}} \mu^{(j+2)}$ such that I is $(\mu_I, \mu^{(j+2)}, 3)$-random.

For given $a \in \text{Field}(\prec)$, let $\mathcal{I}(a)$ be the set of all reals $Z \subseteq \omega$ such that Z is an initial segment of $\prec$, and all elements of Z are bounded by a. $\mathcal{I}(a)$ is a $\Pi^0_1(\mu^{(j)})$ class. Let T_a be a tree recursive in $\mu^{(j)}$ such that $[T_a] = \mathcal{I}(a)$. Given $n \in \omega$, let $T_a\lceil_n = \{\sigma \in T_a \colon |\sigma| = n\}$ be the n-th level of T_a. We have $\mathcal{I}(a) = \bigcap_n [\![T\lceil_n]\!]$.

Now, if $a \in I$, then $\mathcal{I}(a)$ is countable (since in this case each element of $\mathcal{I}(a)$ is an initial segment of the well-founded part of $\prec$ and there are at most countably many such initial segments). Since μ_I is continuous, it follows that $\mathcal{I}(a)$ has μ_I-measure zero.

If, on the other hand, $a \notin I$, then $I \in \mathcal{I}(a)$. Since I is $(\mu_I, \mu^{(j+2)}, 3)$-random and $\mathcal{I}(a)$ is $\Pi^0_1(\mu^{(j)})$, $\mathcal{I}(a)$ does not have μ_I-measure zero: Otherwise we could recursively in $\mu^{(j+2)}$, compute a sequence (l_n) such that $\mu_I[\![T_a\lceil_{l_n}]\!] \leq 2^{-n}$. This would be a $(\mu_I, \mu^{(j+2)}, 1)$-test that covers I, but I is $(\mu_I, \mu^{(j+2)}, 3)$-random.

We obtain the following characterization of I.

$$a \in I \quad \Leftrightarrow \quad \forall n \, \exists l \, (\mu_I[\![T_a\lceil_l]\!] \leq 2^{-n})$$

Since $\mu_I \leq_{\mathrm{T}} \mu^{(j+2)}$, the property on the right hand side is $\Pi^0_2(\mu^{(j+2)})$, hence I is recursive in $\mu^{(j+4)}$, contradicting our initial assumption. □

We conclude this section by establishing that Turing jumps cannot be μ-n-random, $n \geq 2$, for any measure μ. While strictly speaking the following two results are not needed later, they are prototypical for a type of argument that will be important in Section 4, where we construct long sequences of reals with an internal definability hierarchy that are not random with respect to any continuous measure.

Proposition 2.16. *For any $k \geq 0$, if $X \equiv_{\mathrm{T}} \emptyset^{(k)}$, then X is not 2-random with respect to any continuous measure.*

Proof. The case $k = 0$ is clear, so assume $k > 0$. Suppose $X \equiv_{\mathrm{T}} \emptyset^{(k)}$ is μ-2-random for some μ. Then $\emptyset' \leq_{\mathrm{T}} X$ and also $\emptyset' \leq_{\mathrm{T}} \mu'$. It follows from Lemma 2.13 that $\emptyset'$ is recursive in μ. Applying the same argument inductively to $\emptyset^{(i)}$, $i \leq k$, yields $\emptyset^{(i)} \leq_{\mathrm{T}} \mu$, in particular $X \equiv_{\mathrm{T}} \emptyset^{(k)} \leq_{\mathrm{T}} \mu$, which is impossible if X is μ-2-random. □

It may be helpful to picture the preceding argument as a "stair trainer machine": Using the supposedly random X, each step, that is, Turing jump, "sinks down" to μ, and eventually, $X \leq_{\mathrm{T}} \mu$, yielding a contradiction.

The non-randomness property of the jumps extends to infinite jumps, too.

Proposition 2.17. *If $X \equiv_{\mathrm{T}} \emptyset^{(\omega)}$, then X is not 3-random with respect to a continuous measure.*

Proof. Assume for a contradiction that X is μ-3-random for continuous μ. By the inductive argument of the previous proof, $0^{(k)} \leq_{\mathrm{T}} \mu$ for all $k \in \omega$. By a result of Enderton and Putnam [10], if Y is a $\leq_{\mathrm{T}}$-upper bound for $\{0^{(k)} \colon k \in \omega\}$, then $0^{(\omega)} \leq_{\mathrm{T}} Y''$. Therefore, $X \leq_{\mathrm{T}} \mu''$, contradicting that X is μ-3-random. □

3. The Countability Theorem

In this section we will prove Theorem 1, which we restate here for convenience.

Theorem 1. *Let $n \in \omega$. Then the set*

$$\mathrm{NCR}_n = \{X \in 2^\omega : X \text{ is not } n\text{-random for any continuous measure}\}$$

is countable.

As mentioned in the introduction, the case $n = 1$ was proved in [41]. The basic outline for the proof for $n > 1$ is as follows. We first show that the set of reals that are n-random for some continuous measure contains an upper cone in the Turing degrees. The argument uses Martin's result [31] that every Turing invariant Borel set that is cofinal in the partial ordering of the Turing degrees contains an upper cone with respect to Turing reducibility. The base of the cone is given by the winning strategy in a certain Borel game $\mathcal{G}(B)$. The constructive nature of Martin's proof of Borel determinacy yields that winning strategy is contained in a countable level $L_{\beta_{n+3}}$ of the constructible hierarchy. We use a forcing notion due to Kumabe and Slaman (see [46]) to show that given $X \notin L_{\beta_{n+3}}$, there exists a forcing extension $L_{\beta_{n+3}}[G]$ in which every real is Turing reducible to X (relative to the generic G). In particular, X is in the upper cone of random reals above the winning strategy for the game $\mathcal{G}(B)$ (in $L_{\beta_{n+3}}[G]$). Finally, we argue that the winning strategy is absolute and thus makes X random with respect a continuous measure.

3.1. A cone of continuously random reals.

In this section we will prove the existence of an upper cone of random reals.

Definition 3.1. A set $A \subseteq 2^\omega$ is *Turing-invariant* if $X \in A$ and $Y \equiv_{\mathrm{T}} X$ implies that $Y \in A$. A *upper Turing cone* is a Turing invariant set $A \subseteq 2^\omega$

of the form

$$\{Y \in 2^\omega \colon X \leq_{\mathrm{T}} Y\}$$

for some $X \in 2^\omega$.

Borel-Turing determinacy ([31, 32])**.** *If $A \subseteq 2^\omega$ is a Turing invariant Borel set, then there exists a real Y such that either A or $2^\omega \setminus A$ contains an upper Turing cone.*

Lemma 3.2. *For every $n \geq 1$, there exists a real X such that for all $Y \geq_{\mathrm{T}} X$, Y is n-random with respect to some continuous measure.*

Proof. Suppose $X \equiv_{\mathrm{T}(Z)} R$ where R is $(n+2)$-random relative to Z. By Lemma 2.11 X is n-random with respect to some continuous measure.

Let $B \subseteq 2^\omega$ be the set

$$\{Y \in 2^\omega : \exists Z \, \exists R \, (Y \equiv_{\mathrm{T}} Z \oplus R \;\;\&\;\; R \text{ is } (Z, n+2)\text{-random})\}$$

Clearly, B is Turing invariant. To see that B is Borel, note that the set can be defined in the form

$$\exists e, d \; (e, d \text{ are indices of Turing functionals such that } \Phi_d(\Phi_e(Y)) = Y \text{ and one half of } \Phi_e(Y) \text{ is } (n+2)\text{-random relative to the other half}).$$

As observed above, every real in B is n-random for a continuous measure (note that $Y \in B$ cannot be recursive in Z' since R is $(Z, n+2)$-random).

Furthermore, B is *cofinal* in the Turing degrees, i.e., if S is any real, then there exists an $Y \geq_{\mathrm{T}} S$ such that $Y \in B$, since we can always find a real R that is $(n+1)$-random relative to S and put $Y \equiv_{\mathrm{T}} S \oplus R$.

By Borel-Turing determinacy, B contains an upper Turing cone, because B is cofinal in the Turing degrees. □

3.2. **From upper cone to co-countably many.** The determinacy argument of the previous subsection yields the existence of an upper cone of n-random reals. Martin's proof of Borel-Turing determinacy yields that the base of the cone is given by a winning strategy for a certain Borel game.

Although Martin's proof of Borel Determinacy [32] is of a constructive nature, its meta-mathematical complexity is high, in the sense that it makes inductive use of the power set operation: The higher the level of a Borel set, the more iterates of the power set of ω one needs to construct a winning strategy, in form of trees whose nodes are trees whose nodes are trees etc.

Friedman [11] showed that this is in fact an intrinsic feature of Borel determinacy. As we will see in the next section, this supplements Theorem 1 with an interesting metamathematical twist.

Nevertheless, Martin's proof of Borel determinacy is constructive. Therefore, it is not hard to locate a winning strategy within the constructible hierarchy.

Definition 3.3. Given $n \in \omega$, ZFC_n^- denotes the axiom of ZFC, where the power set axiom is replaced by the sentence

"There exist n-many iterates of the power set of ω".

Hence, in ZFC_0^-, for instance, we have the existence of the set of all natural numbers (since the Axiom of Infinity holds and ω is absolute), and various other subsets of ω as given by applications of separation or replacement, but we lack the guaranteed existence of the set of all such subsets.

Models of ZFC_n^- will play an important role throughout this paper. In particular, we are interested in models inside the constructible universe. As usual, L will denote the constructible universe, the limit of the cumulative hierarchy of sets obtained by iterating the power set operation restricted to definable subsets. For any ordinal α, L_α denotes the α-th level of the hierarchy. A key property of this hierarchy is that $|L_\alpha| = |\alpha|$. For more background on L, see [21, Chapter 13] or [28, Chapters V,VI]. For an in-depth account, see [7].

Definition 3.4. Given $n \in \omega$, let β_n be the least ordinal such that

$$L_{\beta_n} \models \mathsf{ZFC}_n^-.$$

By the Löwenheim-Skolem theorem and the Gödel condensation lemma, L_{β_n}, and hence β_n, is countable.

Lemma 3.5. *If $A \subseteq 2^\omega$ is Σ_n^0, then the Borel game $\mathcal{G}(A)$ with winning set A has a winning strategy S in L_{β_n}.*

The proof given by Martin [33] is inductive. A key concept is the *unraveling of a game*. Simply speaking, a tree T over some set B unravels $\mathcal{G}(A)$ if there exists a continuous mapping $\pi\colon [T] \to 2^\omega$ such that $\pi^{-1}(A)$ is clopen in $[T]$, and there is a continuous correspondence between strategies on T and strategies on $2^{<\omega}$.

Martin first shows that Π_1^0 games can be unraveled. The argument is completely constructive, hence can be carried out in L. The unraveling tree

T is given by the legal moves of some auxiliary game whose moves correspond to strategies in the original game on $2^{<\omega}$, that is, reals. To be able to collect all these legal moves requires the existence of the power set of ω.

The inductive step then shows how to unravel a given Σ^0_n set A. Suppose $A = \bigcup A_i$, where each A_i is Π^0_{n-1}. By induction hypothesis, each A_i can be unraveled by some T_i via some mapping π_i. Martin proves that the unravelings T_i can be combined into a single one, T_∞, that unravels each A_i via some π_∞. Since each of the sets $\pi_\infty^{-1}(A_i)$ is clopen, their union $\bigcup \pi_\infty^{-1}(A_i) = \pi_\infty^{-1}(A)$ is open, and can in turn be unraveled by some T. Again, the proof is constructive. The last step in the construction (unraveling $\pi_\infty^{-1}(A)$) passes to a tree of higher type – its nodes correspond to strategies over T_∞. Hence one more iterate of the power set of ω is introduced.

Therefore, Σ^0_n determinacy is provable in ZFC^-_n, and Martin's proof constructs a winning strategy in L_{β_n}, relative to L_{β_n}. By Mostowski's absoluteness theorem (see [21, Theorem 25.4]), this is also a winning strategy in V (see also Subsection 3.4).

In order to complete the proof of Theorem 1, it we need to prove a Posner-Robinson style theorem for reals not contained in L_{β_n} and use an absoluteness argument. We state Lemma 3.6 only in the case that n is greater than zero. Under this restriction, we can avoid class forcing and reduce to standard facts about set forcing. The case n equals zero is not needed for our argument.

Lemma 3.6. *Suppose that n is a natural number greater than zero and X is a real number not in L_{β_n}. Then there exists a model $L_{\beta_n}[G]$ of ZFC^-_n such that every real in $L_{\beta_n}[G]$ is Turing reducible to $X \oplus G$.*

3.3. **Kumabe-Slaman forcing.** This subsection is devoted to proving Lemma 3.6. We construct G by means of a notion of forcing due to Kumabe and Slaman. The forcing was an essential ingredient in the proof of the definability of the Turing jump by Shore and Slaman [46]. It allows for extending the Posner-Robinson Theorem to iterated applications of the Turing jump. It is based on the following partial order. The construction of the generic G follows [46] rather closely. We have to ensure, however, that forcing has the desired set theoretic properties.

In the following, we use the conventions and vocabulary of Section 2.1.

Definition 3.7. Let $\mathbb{P}$ be the following partial order.

(1) The elements p of $\mathbb{P}$ are pairs $(\Phi_p, \vec{Z}_p)$ in which Φ_p is a finite, use-monotone Turing functional and $\vec{Z}_p$ is a finite collection of subsets of ω. As usual, we identify subsets of ω with elements of 2^ω.

(2) If p and q are elements of $\mathbb{P}$, then $p \geq q$ if and only if
   (a) (i) $\Phi_p \subseteq \Phi_q$ and
       (ii) for all $(x_q, y_q, \sigma_q) \in \Phi_q \setminus \Phi_p$ and all $(x_p, y_p, \sigma_p) \in \Phi_p$, the length of σ_q is greater than the length σ_p,
   (b) $\vec{Z}_p \subseteq \vec{Z}_q$,
   (c) for every x, y, and $X \in \vec{Z}_p$, if $\Phi_q(x, X) = y$ then $\Phi_p(x, X) = y$.

In short, a stronger condition than p can add computations to Φ_p, provided that they are longer than any computation in Φ_p and that they do not apply to any element of $\vec{Z}_p$.

Let $\mathbb{P}_n$ denote the partial order $\mathbb{P}$ as defined in L_{β_n}. By standard arguments, we show that if $G \subseteq \mathbb{P}_n$ is a generic filter in the sense of L_{β_n}, then $L_{\beta_n}[G]$ is a model of ZF^-_n. By inspection of $\mathbb{P}_n$, any such G naturally gives rise to a functional $\Phi_G = \bigcup\{\Phi_p : p \in G\}$. To prove Theorem 3.6, given X not in L_{β_n}, we will exhibit a particular G so that G is $\mathbb{P}_n$-generic over L_{β_n} and so that every element in $L_{\beta_n}[G]$ is computable from $G \oplus X$.

Definition 3.8 (Definition III.3.3, [28])**.** Let $\mathbb{P}^*$ be a partially ordered set. Then, $p, q \in \mathbb{P}^*$ are *compatible* iff then have a common extension. An *antichain* is a subset of $\mathbb{P}^*$ whose elements are are pairwise incompatible. $\mathbb{P}^*$ has the *countable chain condition* (ccc) iff, in $\mathbb{P}^*$ every antichain is countable.

Lemma 3.9. *Let n be a natural number greater than zero.*

(1) $\mathbb{P}_n$ is an element of L_{β_n}.
(2) $L_{\beta_n} \models \mathbb{P}_n$ has the ccc.

Proof. Since $n \geq 1$, L_{β_n} satisfies the statement that there is at least one uncountable cardinal. By usual structure theory for initial segments of L, the power set of ω as defined in L_{β_n} is a set in L_{β_n}. Since $\mathbb{P}$ is defined directly from ω, $\mathbb{P}_n$ is a set in L_{β_n}.

If p and q are incompatible elements of $\mathbb{P}_n$, then Φ_p and Φ_q must be different. Since there are only countably many possibilities for Φ_p and Φ_q, any antichain in $\mathbb{P}_n$ must be countable in L_{β_n}. □

Definition 3.10. Let G be a subset of $\mathbb{P}_n$.

(1) G is a *filter* on $\mathbb{P}_n$ iff
   (a) G is not empty.

(b) $\forall p, q \in G\, \exists r \in G[p \geq r$ and $q \geq r]$.
(c) $\forall p, q \in G[$if $p \geq q$ and $q \in G,$ then $p \in G]$.

(2) G is $\mathbb{P}_n$-*generic over* L_{β_n} iff for all D such that $D \subseteq \mathbb{P}_n$ is dense and D is definable from parameters in L_{β_n}, $G \cap D$ is not empty. Here, D is dense iff for every $p \in \mathbb{P}_n$ there is a $q \in D$ such that $p \geq q$.

Lemma 3.11. *If G is $\mathbb{P}_n$-generic over L_{β_n}, then $L_{\beta_n}[G] \models \mathsf{ZFC}_n^-$.*

Proof. By Lemma IV.2.26 of [28], it follows that if G is $\mathbb{P}_n$ generic over L_{β_n}, then $L_{\beta_n}[G]$ satisfies the axioms of ZF^- except for possibly Replacement. It remains only to observe that $L_{\beta_n}[G]$ satisfies Replacement, Choice and that there are n-many uncountable cardinals. The verification that $L_{\beta_n}[G]$ satisfies Replacement is the same as given in Theorem IV.2.27 of [28]. That $L_{\beta_n}[G]$ satisfies Choice follows from Replacement and the usual proof that the order of constructibility is a L_{β_n}-definable well-order of L_{β_n} applies relative to G. Finally, that $L_{\beta_n}[G]$ has the same uncountable cardinals as L_{β_n} does follows from $\mathbb{P}_n$'s having the ccc in L_{β_n} by the argument given in the proof of Theorem IV.3.4 of [28]. Although this theorem is stated for models of ZFC, its proof does not invoke the power set axiom. Given that $L_{\beta_n}[G]$ has n many uncountable cardinals, the Gödel Condensation Lemma relative to G ensures that it has n-many iterates of the power set of ω. □

Next, we show that every dense set in L_{β_n} can be met via an extension adding no computations along X. This is crucial for the construction in [46].

Lemma 3.12. *Let $D \in L_{\beta_n}$ be dense in $\mathbb{P}_n$ and suppose $X \notin L_{\beta_n}$. For any $p \in \mathbb{P}_n$, there exists a $q \leq p$ such that $q \in D$ and Φ_q does not add any new computation along X.*

Proof. Suppose $p = (\Phi_p, \vec{Z}_p)$ is in $\mathbb{P}_n$. We say a string τ is *essential* for (p, D) if, whenever $q < p$ and $q \in D$, there exists a triple $(x, y, \sigma) \in \Phi_q \setminus \Phi_p$ such that σ is compatible with τ. In other words, whenever one meets D by an extension of p, a computation relative to some string compatible with τ is added. Note that τ's being essential for (p, D) is definable in L_{β_n}.

If τ is a binary sequence and τ_0 is an initial segment of τ, then any sequence σ which is compatible with τ is also compatible with τ_0. Thus, being essential is closed under taking initial segments. So,

$$T(p, D) = \{\tau \colon \tau \text{ essential for } (p, D)\ \}$$

is a binary tree in L_{β_n}.

Assume now for a contradiction that a q as postulated above does not exist. This means that for any $r \leq p$, either $r \notin D$ or Φ_r adds a computation along X. It follows that every initial segment $\tau \subset X$ is essential for (p, D). Thus $T(p, D)$ is infinite. Since L_{β_n} satisfies Kőnig's Lemma (equivalently, compactness of the Cantor set), there exists a real $Y \in 2^\omega \cap L_{\beta_n}$ such that Y is an infinite path through $T(p, D)$.

Now consider the condition $p_1 = (\Phi_p, \vec{Z}_p \cup \{Y\})$. As $\Phi_q = \Phi_p$ and $Y \in L_{\beta_n}$, we trivially have $p_1 \leq p$ in $\mathbb{P}_n$. Since every initial segment of Y is essential for (p, D), any extension of p in D must add a computation along Y. Since no extension of p_1 can add a computation along Y and every extension of p_1 is an extension of p, no extension of p_1 is in D. This contradicts the density of D. □

We can now finish the proof of Theorem 3.6. It is sufficient to construct a $\mathbb{P}_n$-filter G that is generic over L_{β_n} such that for every $A : \omega \to 2$ in $L_{\beta_n}[G]$ there is a k such that for all m, $\Phi_G^X((k, m)) = A(m)$. The construction of such a G follows [46]. We fix countings of the set of terms in the forcing language for functions from ω to 2 in L_{β_n} and of the dense subsets D of $\mathbb{P}_n$ which are definable over L_{β_n}. We proceed by recursion to define G. At stage n, we will have determined for each i less than n, an integer k_i and we will ensure that for all m, $\Phi_G^X((k_i, m))$ will have the same value as the interpretation of the i-th term does at m in L_{β_n}. By Lemma 3.12, we can meet dense sets and decide values of terms without adding any new values to Φ_G^X. We can then extend Φ_G so that $\Phi_G^X((k_i, m))$ takes the values equal to those already decided for the relevant terms. Finally, we can determine a value for k_n that is greater than any argument for any computation mentioned in the construction so far.

We conclude by observing that the set Φ_G satisfies the two conclusions of Theorem 3.6. First, $L_{\beta_n}[\Phi_G]$ is model of ZFC_n^- since $\Phi_G \in L_{\beta_n}[G]$ and $L_{\beta_n}[G]$ is a model of ZFC_n^-. Second, for each $Z : \omega \to 2$, if $Z \in L_{\beta_n}[\Phi_G]$ then it is the denotation in $L[G]$ of some term in the forcing language for $\mathbb{P}_n$, and so there is a k such that for all n, $Z(n) = \Phi_G^X((k, n))$.

3.4. **Completing the proof of Theorem 1.** We now put the pieces together to show that every real outside of $L_{\beta_{n+3}}$ is n-random with respect to a continuous probability measure. As $L_{\beta_{n+3}}$ is countable, this will complete the proof of Theorem 1.

Given $X \notin L_{\beta_{n+3}}$, choose G as in Lemma 3.6. Consider the game defined in the proof of Lemma 3.2. Its winning set is

$$B = \{X \in 2^\omega : \exists Z \, \exists R \, (X \equiv_{\mathrm{T}} Z \oplus R \ \& \ R \text{ is } (Z, n+2)\text{-random})\},$$

which is Σ^0_{n+3}.

Now relativize to G and denote the relativized winning set by B^G. Since ZFC^-_{n+3} proves Σ^0_{n+3} determinacy, $L_{\beta_{n+3}}[G]$ contains a winning strategy for the relativized game $\mathcal{G}(B^G)$ played inside $L_{\beta_{n+3}}[G]$.

The property of being a winning strategy for a given Borel game is $\mathbf{\Pi}^1_1$. By Mostowski's absoluteness theorem (see [21, Theorem 25.4]), this means that a winning strategy in $L_{\beta_{n+3}}[G]$ is actually a winning strategy "in the real world", i.e., it wins on *all* plays, not just the ones in $L_{\beta_{n+3}}[G]$.

Hence, following the proof of Lemma 3.2, *every* real in the upper cone (relative to G) of the winning strategy is (μ, G, n)-random for some continuous measure μ. By Lemma 3.6, X is (relative to G) in every upper cone with base in $L_{\beta_{n+3}}[G]$, X (μ, G, n)-random for some continuous measure μ. Finally, note that every (μ, G, n)-random real is (μ, n)-random.

We have shown that every real not contained in $L_{\beta_{n+3}}$ is n-random for a continuous measure. As β_{n+3} is countable, this completes the proof of Theorem 1.

4. The Metamathematics of Randomness

In this section, we will show that the metamathematical ingredients used to prove the countability of NCR_n are necessary. More precisely, we will prove the Theorem 2, which we restate here for convenience.

Theorem 2. *There exists a computable function $G(n)$ such that for every $n \in \omega$,*

$$\mathsf{ZFC}^-_n \nvdash \text{“}\mathrm{NCR}_{G(n)} \textit{ is countable.”}$$

Before starting the proof, we outline its basic idea. For given n, we will show that in the model L_{β_n} of ZFC^-_n, $\mathrm{NCR}_{G(n)}$ is not countable. To this end, we find a sequence (Y_α) of reals that satisfies

(1) (Y_α) is cofinal in the Turing degrees of L_{β_n}, hence not countable in L_{β_n},
(2) no Y_α is $G(n)$-random for a continuous measure in L_{β_n}.

As we have seen in Propositions 2.16 and 2.17, iterating the Turing produces an increasing sequence of non-random reals. It makes sense therefore to look for a set-theoretic analogue of the jump hierarchy. This analogue is given by the *master codes* of the constructible hierarchy. Just as instances of the jump code levels of arithmetically definable subsets of ω, master codes code levels of the constructible universe. The master codes for a higher level of L can be obtained from codes for lower levels by iterating definability. This will be crucial for our proof, since it allows for applying the "Stair Trainer"-argument of Propositions 2.16 and 2.17 in this setting.

In order to define this sequence, we have to present a few more facts on L. In particular, we are interested how at each step new sets are added to L. This is the heart of the *fine structure theory* of L due to Jensen [22]. We will give a brief review of the core concepts and results. Readers familiar with fine structure theory can skip ahead to Subsection 4.3.

4.1. **Fine structure and Jensen's J-hierarchy.** Fine structure provides a level-by-level, quantifier-by-quantifier analysis of how new sets are generated in L. Jensen defines the new constructible hierarchy, the *J-hierarchy* $(J_\alpha)_{\alpha\in\mathrm{Ord}}$ that has all the important properties of the L-hierarchy (in particular, $L = \bigcup_\alpha J_\alpha$). In addition to this, each level J_α has closure properties (such as under pairing functions) that L_α may be lacking. While it is not strictly necessary for this paper to work with J_α (we could work with $(L_{\omega\alpha})_{\alpha\in\mathrm{Ord}}$), the J-hierarchy is the established framework for fine structure analysis, and we will adopt its basic concepts and terminology.

The sets J_α are obtained by closing under a scheme of *rudimentary functions*. In contrast to $L_{\alpha+1}$, $J_{\alpha+1}$ contains sets of rank up to $\omega(\alpha+1)$, not just subsets of J_α, e.g. ordered pairs. The rudimentary functions are essentially a scheme of *primitive set recursion* [23].

For transitive X, $\mathrm{rud}(X)$ denotes the smallest set Y that contains $X\cup\{X\}$ and is closed under rudimentary functions (rud closed). The inclusion of $\{X\}$ when taking the rudimentary closure guarantees that new sets are introduced even if X is closed under rudimentary functions.

The J-hierarchy is introduced as a cumulative hierarchy induced by the rud-operation:

$$\begin{aligned} J_0 &= \emptyset \\ J_{\alpha+1} &= \mathrm{rud}(J_\alpha) \\ J_\lambda &= \bigcup_{\alpha<\lambda} J_\alpha \quad \text{for } \lambda \text{ limit.} \end{aligned}$$

A fine analysis of the rudimentary functions reveals that the rud-operation can be completed by iterating some or all of *nine basic rudimentary functions.*

Proposition 4.1 (Jensen [22])**.** *Every rudimentary function is a composition of the following nine functions:*

$$
\begin{aligned}
F_0(x,y) &= \{x,y\},\\
F_1(x,y) &= x \setminus y,\\
F_2(x,y) &= x \times y,\\
F_3(x,y) &= \{(u,z,v)\colon z \in x \ \wedge \ (u,v) \in y\},\\
F_4(x,y) &= \{(u,v,z)\colon z \in x \ \wedge \ (u,v) \in y\},\\
F_5(x,y) &= \bigcup x,\\
F_6(x,y) &= \operatorname{dom}(x),\\
F_7(x,y) &= \in \cap \, (x \times x),\\
F_8(x,y) &= \{\{x(z)\}\colon z \in y\}.
\end{aligned}
$$

The S-operator is defined as taking a one-step application of any of the basic functions,

$$
S(X) = [X \cup \{X\}] \cup \left[\bigcup_{i=0}^{8} F_i[X \cup \{X\}] \right]. \tag{4.1}
$$

For transitive X, it holds that [22, Corollary 1.10]

$$
\operatorname{rud}(X) = \bigcup_{n \in \omega} S^{(n)}(X).
$$

The S-hierarchy is defined as the cumulative hierarchy induced by the S-operator and refines the J-hierarchy.

$$
\begin{aligned}
S_0 &= \emptyset,\\
S_{\alpha+1} &= S(S_\alpha),\\
S_\lambda &= \bigcup_{\alpha<\lambda} S_\alpha \qquad \text{for } \lambda \text{ limit.}
\end{aligned}
$$

We obviously have

$$
J_\alpha = \bigcup_{\beta < \omega\alpha} S_\beta = S_{\omega\alpha}.
$$

We list a few basic properties of the sets J_α. For details and proofs, see [22] or [7].

- Each J_α is transitive and and is a model of a sufficiently large fragment of set theory (more precisely, it is a model of KP-set theory without Σ_0-collection).
- The hierarchy is *cumulative*, i.e., $\alpha \leq \beta$ implies $J_\alpha \subseteq J_\beta$.
- $\text{rank}(J_{\alpha+1}) = \text{rank}(J_\alpha) + \omega$. Each successor step adds ω new ordinals. $J_\alpha \cap \text{Ord} = \omega\alpha$, in particular, $J_1 = V_\omega$ and $J_1 \cap \text{Ord} = \omega$.
- $(J_\alpha)_{\alpha \in \text{Ord}}$ and $(L_\alpha)_{\alpha \in \text{Ord}}$ generate the same universe: $L = \bigcup_\alpha J_\alpha$. Moreover, $L_\alpha \subseteq J_\alpha \subseteq L_{\omega\alpha}$, and $J_\alpha = L_\alpha$ if and only if $\omega\alpha = \alpha$. Finally, $J_{\alpha+1} \cap \mathcal{P}(J_\alpha) = \mathcal{P}_{\text{DEF}}(J_\alpha)$, that is, $J_{\alpha+1}$ contains precisely those subsets of J_α that are first order definable over J_α.
- The Σ_n-satisfaction relation over J_α, $\models^{\Sigma_n}_{J_\alpha}$, is Σ_n-definable over J_α, uniformly in α.
- The mapping $\beta \mapsto J_\beta$ $(\beta < \alpha)$ is Σ_1-definable over any J_α.
- There is a Π_2 formula $\varphi_{\mathsf{V=L}}$ such that for any transitive set M,

$$M \models \varphi_{\mathsf{V=L}} \Leftrightarrow \exists \alpha\, M = J_\alpha.$$

The J-hierarchy shares all important metamathematical features with the L-hierarchy. We cite the two most important facts. The L-versions of the two propositions together constitute the core of Gödel's proof that GCH and AC hold in L.

Proposition 4.2. *There exists a Σ_1-definable well-ordering $<_J$ of L and for any $\alpha > 1$, the restriction of $<_J$ to J_α is uniformly Σ_1-definable over J_α.*

Proposition 4.3 (The condensation lemma for J)**.** *For any α, if $X \preceq_{\Sigma_1} J_\alpha$, then there is an ordinal β and an isomorphism π between X and J_β. Both β and π are uniquely determined.*

For proofs of these results for the J-hierarchy again refer to [22] or [7].

4.2. **Projecta and master codes.** The definable well-ordering $<_J$ together with the definability of the satisfaction relation can be used to show that each J_α has *definable Skolem functions*, essentially by selecting the $<_J$-least witness that satisfies an existential formula. The definable Skolem functions can in turn be used to define a canonical indexing of J_α [22, Lemma 2.10].

Proposition 4.4 (Jensen [22])**.** *For each α, there exists a $\Sigma_1(J_\alpha)$-definable surjection from $\omega\alpha$ onto J_α.*

While a simple cardinality argument yields that $|J_\alpha| = |\omega\alpha|$, Jensen's result shows that an $\omega\alpha$-counting of J_α already exists in $J_{\alpha+1}$. The indexing is obtained by taking (essentially) the Skolem hull of $\omega\alpha$ under the canonical Σ_1-Skolem function. The resulting set X is a Σ_1-elementary substructure of J_α, hence by the condensation lemma is isomorphic to some J_β. The isomorphism taking X to J_β is the identity on all ordinals below $\omega\alpha$, and one can show that this in turn implies that the isomorphism must be the identity on X, i.e., $X = J_\beta$.

Boolos and Putnam [3] first observed that if a new real is defined in $L_{\alpha+1}$, i.e., if

$$\mathcal{P}(\omega) \cap (L_{\alpha+1} \setminus L_\alpha) \neq \emptyset,$$

then the strong absoluteness properties of L can be used to get a *definable* ω-counting of L_α (instead of just an α-counting as above). Because, if a new subset Z of ω is constructed in $L_{\alpha+1} \setminus L_\alpha$, one can take the Skolem hull of ω instead of $\omega\alpha$. The resulting $X \cong L_\beta$ is still equal to L_α, since the definition of the new real applies in the elementary substructure L_β. If $\beta < \alpha$, then this would contradict the fact that $Z \notin L_\alpha$.

Proposition 4.5 (Boolos and Putnam [3])**.** *If $\mathcal{P}(\omega) \cap (L_{\alpha+1} \setminus L_\alpha) \neq \emptyset$, then there exists a surjection $f : \omega \to L_\alpha$ in $L_{\alpha+1}$.*

Of course, at some stages no new reals are constructed. Boolos and Putnam [3] showed that the first such stage is precisely the ordinal β_0, i.e., the least ordinal β such that $L_\beta \models \mathsf{ZF}^-$. By Gödel's work, on the other hand, we know that no new real is constructed after stage ω_1^L.

Jensen [22] vastly extended these ideas into the framework of *projecta* and *master codes*, which form the core concepts of *fine structure theory*.

Definition 4.6. For natural numbers $n > 0$ and ordinals $\alpha > 0$, the Σ_n-*projectum* ρ_α^n is equal to the least $\gamma \leq \alpha$ such that $\mathcal{P}(\omega\gamma) \cap (\Sigma_n(J_\alpha) \setminus J_\alpha) \neq \emptyset$.

We put $\rho_\alpha^0 = \alpha$. Hence $1 \leq \rho_\alpha^n \leq \alpha$ for all n. As ρ_α^n is non-increasing in n, we can also define

$$\rho_\alpha = \min_n \rho_\alpha^n \quad \text{ and } \quad n_\alpha = \min\{k \colon \rho_\alpha^k = \rho_\alpha\}.$$

Jensen [22, Theorem 3.2] proved that the projectum ρ_α^n is equal to the least $\delta \leq \alpha$ such that there exists a function f that is $\Sigma_n(J_\alpha)$-definable over J_α such that $f(D) = J_\alpha$ for some $D \subseteq \omega\delta$, establishing the analogy with the Boolos-Putnam result. From this it follows that if $\rho_\alpha^n < \alpha$, it must be a cardinal in J_α, for all n.

Jensen gave another characterization of the projectum, which in fact he used as his original definition in [22]. Suppose $\langle M, \in\rangle$ is a set-theoretic structure. We can extend this structure by adding an additional relation $A \subset M$. If we do this, we would like the structure to satisfy some basic set theoretic closure properties. For instance, we would like our universe to satisfy the comprehension axiom with respect to the new relation, that is, whenever we pick an $x \in M$, the collection of elements in x that satisfy A should be in M. Such structures are called *amenable*.

Definition 4.7. Given $A \subseteq M$, the structure $\langle M, A \rangle$ is called *amenable*, if M is an amenable set and

$$\forall x \in M \; [x \cap A \in M].$$

Jensen [22, Theorem 3.2] showed that

$$\begin{aligned} \rho^n_\alpha = {} & \text{the largest ordinal } \gamma \leq \alpha \text{ such that} \\ & \quad \langle J_\gamma, A \rangle \text{ is amenable for any } A \subseteq J_\gamma \text{ that is in } \Sigma_n(J_\alpha). \end{aligned}$$

This means the projectum ρ^n_α identifies the "stable" core of J_α with respect to Σ_n definability over J_α.

Being amenable with rud-closed domain can also be characterized via relative rud-closedness. This will be important later.

Definition 4.8 (Jensen [22])**.** A function f is *A-rud* if it can be obtained as a combination of the basis functions $F_1, \dots, F_8$ and the function

$$F_A(x, y) = x \cap A.$$

A structure $\langle M, A \rangle$ is *rud closed* if $f[M^n] \subseteq M$ for all A-rud functions f.

Proposition 4.9 (Jensen [22])**.** *A structure $\langle M, A \rangle$, $A \subseteq M$, is rud closed if and only if M is rud closed and $\langle M, A \rangle$ is amenable.*

The existence of a definable surjection between (a subset of) $\omega\rho^n_\alpha$ and $\Sigma_n(J_\alpha)$ allows for coding $\Sigma_n(J_\alpha)$ into its projectum. One way this can be implemented is via so-called *master codes*.

Definition 4.10. A Σ_n *master code* for J_α is a set $A \subseteq J_{\rho^n_\alpha}$ that is $\Sigma_n(J_\alpha)$, such that for any $m \geq 1$,

$$\Sigma_{n+m}(J_\alpha) \cap \mathcal{P}(J_{\rho^n_\alpha}) = \Sigma_m(\langle J_{\rho^n_\alpha}, A \rangle).$$

A Σ_n master code does two things:

(1) It "accelerates" definitions of new subsets of $J_{\rho^n_\alpha}$ by n quantifiers.

(2) It replaces parameters from J_α in the definition of these new sets by parameters from $J_{\rho^n_\alpha}$ (and the use of A as an "oracle").

The existence of master codes follows rather easily from the existence of a $\Sigma_n(J_\alpha)$-mapping from $\omega\rho^n_\alpha$ onto J_α. However, for $n > 1$, this mapping is *not uniform*. Jensen exhibited a uniform, canonical way to define master codes, by iterating Σ_1-definability.

Put

$$A^0_\alpha = \emptyset, \quad p^0_\alpha = \emptyset.$$

Assuming that A^n_α is a Σ_n master code, it is not hard to see that every set $x \in J_{\rho^n_\alpha}$ is Σ_1-definable over $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$ with parameters from $J_{\rho^{n+1}_\alpha}$ and one parameter from $J_{\rho^n_\alpha}$ (used to define a surjection from $\omega\rho^{n+1}_\alpha$ onto $J_{\rho^n_\alpha}$). Hence we can put

$$\begin{aligned} p^{n+1}_\alpha = {} & \text{the } <_J\text{-least } p \in J_{\rho^n_\alpha} \text{ such that every } u \in J_{\rho^n_\alpha} \text{ is } \Sigma_1 \text{ definable} \\ & \text{over } \langle J_{\rho^n_\alpha}, A^n_\alpha\rangle \text{ with parameters from } J_{\rho^{n+1}_\alpha} \cup \{p\}\ . \end{aligned}$$

The p^n_α are called the *standard parameters*.

Using p^{n+1}_α, we can code the Σ_1 elementary diagram of the structure $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$ into a set A^{n+1}_α:

$$A^{n+1}_\alpha := \{(i,x)\colon i \in \omega \ \wedge\ x \in J_{\rho^{n+1}_\alpha} \ \wedge\ \langle J_{\rho^n_\alpha}, A^n_\alpha\rangle \models \varphi^{(2)}_i(x, p^{n+1}_\alpha)\},$$

where $(\varphi^{(k)}_i)$ is a standard Gödel numbering of all Σ_1 formulas with k free variables. It is not hard to verify that A^{n+1}_α is a Σ_{n+1} master code for J_α. Furthermore, the structure $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$ is amenable for each $\alpha > 1$, $n \geq 0$. We will call the structure $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$ the *standard* Σ_n *J-structure* for J_α.

Definition 4.11. We denote the standard J-structure over J_α at the 'ultimate' projectum n_α by

$$\langle J_{\rho_\alpha}, A_\alpha\rangle := \langle J_{\rho^{n_\alpha}_\alpha}, A^{n_\alpha}_\alpha\rangle.$$

One consequence of the A^n_α being master codes is that we can obtain the sequence of projecta of an ordinal by iterating taking Σ_1-projecta relative to $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$. Given an amenable structure $\langle J_\alpha, A\rangle$, the *$\Sigma_n$-projectum* $\rho^n_{\alpha,A}$ of $\langle J_\alpha, A\rangle$ is defined to be the largest ordinal $\rho \leq \alpha$ such that $\langle J_\rho, B\rangle$ is amenable for any $B \subseteq J_\rho$ that is in $\Sigma_n(\langle J_\alpha, A\rangle)$.

Proposition 4.12 (Jensen [22])**.** *For $\alpha > 1, n \geq 0$,*

$$\rho^{n+1}_\alpha = \rho^1_{\rho^n_\alpha, A^n_\alpha}.$$

In particular, the standard Σ_{n+1} J-structure for $J_\alpha = \langle J_\alpha, \varnothing\rangle$ is the standard Σ_1 J-structure for $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$.

4.3. **ω-Copies of J-structures.** We later want to apply the recursion theoretic techniques of Section 2 to countable J-structures. We therefore have to code them as subsets of ω. If the projectum ρ^n_α is equal to 1, all set-theoretic information about the J-structure $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$ is contained in the master code A^n_α, which is simply a real, and hence lends itself directly to recursion theoretic analysis. Starting with the work by Boolos and Putnam [3], this has been studied in a number of papers (e.g. [24], [19]).

In this subsection we give a recursion theoretic analysis of the internal workings of a countable presentation of a J-structure.

Definition 4.13. Let $X \subseteq \omega$. The *relational structure* induced by X is $\langle F_X, E_X\rangle$, where

$$xE_Xy \Leftrightarrow \langle x, y\rangle \in X$$

and

$$F_X = \mathrm{Field}(E_X) = \{x \colon \exists y\, (xE_Xy \text{ or } yE_Xx)\}.$$

The idea is that a number x represents a code for the set whose codes are the numbers y with yE_Xx,

$$\mathrm{Set}_X(x) = \{y \colon yE_Xx\}.$$

The relational structure $\langle F_X, E_X\rangle$ is *extensional* if

$$\forall x, y \in F_X\, [(\forall z\, zE_Xx \Leftrightarrow zE_Xy) \ \Rightarrow x = y],$$

that is

$$\forall x, y \in F_X\, (x \neq y \Rightarrow \mathrm{Set}_X(x) \neq \mathrm{Set}_X(y)).$$

Mostowski's Collapsing Theorem states that if $\langle F_X, E_X\rangle$ is extensional and well-founded, it is isomorphic to a unique structure $(M, \in)$, where M is a transitive set. In this sense we can speak of a countable set theoretic structure *coded* by X. If $\varphi(v_1, \dots, v_n)$ is a formula in the language of the set theory, we can interpret it over $\langle F_X, E_X\rangle$ and write

$$X \models \varphi[a_1, \dots, a_n]$$

for $\langle F_X, E_X\rangle \models \varphi[a_1, \dots, a_n]$ with $a_i \in F_X$.

J-structures have an additional set A, and we capture this on the coding side via pairs $\langle X, M\rangle$, where $M \subseteq F_X$. Semantically, A and M are seen as interpreting a predicate added to the language. This way we can consider the satisfaction relation $\langle X, M\rangle \models \varphi$, where φ is a set-theoretic formula with an additional unary predicate.

We are particularly interested in relational structures that code countable standard J-structures. The following is a generalization of the definition due to Boolos and Putnam [3]

Definition 4.14. An *ω-copy* of a countable, extensional, set-theoretic structure $\langle S, A\rangle$, $A \subseteq S$, is a pair $\langle X, M\rangle$ of subsets of ω such that X codes the structure $\langle F_X, E_X\rangle$ in the sense of Definition 4.13, and such that there exists a bijection $\pi : S \to F_X$ such that

$$\forall x, y \in S\,[\, x \in y \iff \pi(x) E_X \pi(y)\,], \tag{4.2}$$

and

$$M = \{\pi(x) \colon x \in A\}. \tag{4.3}$$

The definition thus means an ω-copy $\langle X, M\rangle$ of $\langle S, A\rangle$ is isomorphic to $\langle S, A\rangle$ when seen as structures over the language of set theory.

If $A = \varnothing$, then necessarily $M = \varnothing$, and in this case we say X is an ω-copy of S.

We will now consider ω-copies of standard J-structures.

If $\rho^n_\alpha = 1$, we have $J_{\rho^n_\alpha} = L_\omega = V_\omega$, i.e., the *hereditarily finite sets*. In this case, we obtain an ω-copy by fixing a bijection between ω and V_ω, c.f. [1]. We let $\langle x, y\rangle \in X$, i.e., $x\, E_X\, y$, if and only if $\pi_\omega(x) \in \pi_\omega(y)$ and $x \in M$ if and only if $\pi_\omega(x) \in A^{n+1}_\alpha$. Then $\langle X, M\rangle$ is an ω-copy of $\langle J_{\rho^{n+1}_\alpha}, A^{n+1}_\alpha\rangle$ via π^{-1}_ω.

Definition 4.15. When $\rho^n_\alpha = 1$, the *canonical copy* of $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$ is the ω-copy defined above.

We will now show that from a canonical copy of $\langle J_{\rho^{n+1}_\alpha}, A^{n+1}_\alpha\rangle$, we can extract ω-copies of all $\langle J_{\rho^i_\alpha}, A^i_\alpha\rangle$, $i \leq n$, in an effective and uniform way.

By choice of p^{n+1}_α, for every $u \in J_{\rho^n_\alpha}$, there exists a Σ_1-formula $\psi(v_0, v_1, v_2)$ and $x \in J_{\rho^{n+1}_\alpha}$ such that u is the only solution over $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$ to $\psi(v_0, x, p^{n+1}_\alpha)$.

Definition 4.16. A pair (i, x), $i \in \omega, x \in J_{\rho^{n+1}_\alpha}$ is an *n-code* if there exists a $u \in J_{\rho^n_\alpha}$ such that u is the unique solution to

$$\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle \models \varphi^{(3)}_i(v_0, x, p^{n+1}_\alpha).$$

(Recall that $(\varphi_i^{(k)})$ is a standard Gödel numbering of all Σ_1 formulas with k free variables.)

We can check the property of being an n-code using Σ_1 formulas: (i, x) is an n-code if and only if

$$\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle \models \exists v_0\, \psi_i(v_0, x, p_\alpha^{n+1}) \tag{4.4}$$

and

$$\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle \not\models \exists v_0, v_1\, (\psi_i(v_0, x, p_\alpha^{n+1}) \wedge \psi_i(v_1, x, p_\alpha^{n+1}) \wedge v_0 \neq v_1). \tag{4.5}$$

This means a standard code has the information necessary to sort out n-codes among its elements. Relative to a canonical copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1} \rangle$, it is decidable whether some number is the image of an n-code, due to the effective way we translate between finite sets and their codes.

If $\langle X, M \rangle$ is an arbitrary ω-copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1} \rangle$ via π, and $(i, x) \in J_{\rho_\alpha^{n+1}}$ is an n-code, then we call $\pi((i, x))$ a π-n-code. Being able to decide relative to $\langle X, M \rangle$ whether a number is a π-n-code of a pair hinges on knowledge of the following two functions

(i) the mapping $n_X : n \mapsto \pi(n)$ $(n \in \omega)$,
(ii) the mapping $h_X : (\pi(x), \pi(y)) \mapsto \pi((x, y))$.

These mappings may not be computable relative to $\langle X, M \rangle$, but they are definable, as follows.

If $\alpha > 1$, then $\omega \in J_\alpha$, and an ω-copy of any such J_α must contain a witness for ω. In this case, we can recover n_X recursively in X'.

Lemma 4.17. *If X is an ω-copy of J_α, $\alpha > 1$, then n_X is computable in X'.*

Proof. We approximate $n_X(i)$ from below. Let $z = \pi(\omega)$. At stage 0, put $n_{X,0}(i) = 0$ for all i. At stage s, we test whether $s E_X z$. If yes, we can determine how s relates to the previous elements of z discovered, that is, we can compute the finite linear order of the elements of z seen so far, say $n_0 E_X \dots E_X n_k$. We put $n_{X,s}(i) = n_i$ for $i \leq k$. The assertion now follows from the Limit Lemma. □

This argument applies more generally as follows.

Lemma 4.18. *If $\langle X, M \rangle$ is an ω-copy of $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$, and $\rho_\alpha^n = 1$, then $\langle X, M \rangle'$ uniformly computes the canonical copy of $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$.*

Proof. Suppose $\langle X, M\rangle$ is an ω-copy of $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$ via π. We need to show that the mapping $\pi_\omega \circ \pi^{-1}$ is recursive in $\langle X, M\rangle'$. The map $n_{\langle X,M\rangle}$ is recursive in $\langle X, M\rangle'$ and therefore $\langle X, M\rangle'$ can compute the isomorphism between M and the image of A^n_α in the canonical copy. □

We can also recover the function h_X arithmetically in X.

Lemma 4.19. *If X is an ω-copy of J_α, then the function h_X is computable in $X^{(2)}$.*

Proof. We have

$$\begin{aligned} h_X(\pi(x), \pi(y)) = b \;\Leftrightarrow\; \exists c, d \,\big[&\forall z\,(z E_X c \;\Leftrightarrow z = \pi(x)) \\ &\wedge\; \forall z\,(z E_X d \Leftrightarrow z = \pi(x) \;\vee\; z = \pi(y)) \\ &\wedge\; \forall z\,(z E_X b \;\Leftrightarrow\; z = c \;\vee\; z = d)\big] \end{aligned}$$

□

Definition 4.20. Suppose $\langle X, M\rangle$ is an ω-copy via π of a rud closed structure $\langle J, A\rangle$. We say $\langle X, M\rangle$ is *effective* if the functions n_X and h_X are recursive in $X \oplus M$.

Lemma 4.21. *If $\rho^n_\alpha = 1$, the canonical copy of $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$ is effective.*

Proof. The mapping π_ω^{-1} satisfies the conditions required in Definition 4.20 naturally. □

Lemma 4.22. *If $\langle X, M\rangle$ is an effective copy of $\langle J_{\rho^{n+1}_\alpha}, A^{n+1}_\alpha\rangle$ via π, then the mapping*

$$\pi((u, v)) \mapsto (\pi(u), \pi(v)),$$

where $u, v \in J_{\rho^{n+1}_\alpha}$, is recursive in $X \oplus M$. The mapping

$$\pi((i, u)) \mapsto (i, \pi(u)),$$

where $i \in \omega$, is also recursive in $X \oplus M$.

Proof. The first mapping can be computed by inverting h_X (which must be one-one), the second mapping by additionally inverting n_X. □

Lemma 4.23. *If $\langle X, M\rangle$ is an effective copy of $\langle J_{\rho^{n+1}_\alpha}, A^{n+1}_\alpha\rangle$ via π, then it is decidable in $X \oplus M$ whether a number $y \in \omega$ is a π-n-code.*

Proof. Suppose $y \in M$ (if not, it cannot be a π-n-code). Then $y = \pi((i, x))$ for some $(i, x) \in A^{n+1}_\alpha$. Since the copy is effective, we have $\pi((i, x)) = h_X(\pi(i), \pi(x))$, and by Lemma 4.22 we can find i and $\pi(x)$ recursively in $X \oplus M$.

Recall that $(\varphi_i^{(2)})$ is a standard Gödel numbering of the Σ_1 formulas with two free variables. There exist recursive functions g_1, g_2 such that $\varphi_{g_1(i)}^{(2)}$ and $\varphi_{g_2(i)}^{(2)}$ are Σ_1 formulas equivalent (over $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$) to the formulas in (4.4) and (4.5), respectively. Then (i, x) is a n-code if and only if $(g_1(i), x) \in A_\alpha^{n+1}$ and $(g_2(i), x) \notin A_\alpha^{n+1}$.

By the effectiveness of the ω-copy, the latter two conditions are equivalent to

$$h_X(\pi(g_1(i)), \pi(x)) \in M \text{ and } h_X(\pi(g_2(i)), \pi(x)) \notin M,$$

which is recursive in $X \oplus M$. □

Two n-codes (i_0, x_0) and (i_1, x_1) *represent the same set* $u \in J_{\rho_\alpha^n}$ if u is the unique solution to $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle \models \varphi_{i_0}^{(3)}(v_0, x_0, p_\alpha^{n+1})$ and $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle \models \varphi_{i_1}^{(3)}(v_0, x_1, p_\alpha^{n+1})$. A similar property can be defined for π-n-codes.

Lemma 4.24. *If $\langle X, M \rangle$ is an effective copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1} \rangle$ via π, then it is decidable in $X \oplus M$ whether two numbers are π-n-codes of the same set.*

Proof. (i_0, x_0) and (i_1, x_1) represent different sets if and only if

$$\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle \models \exists v_0, v_1, v_2 \left[\varphi_{i_0}^{(3)}(v_0, x_0, p_\alpha^{n+1}) \wedge \varphi_{i_1}^{(3)}(v_1, x_1, p_\alpha^{n+1}) \wedge \right. \\ \left. (v_2 \in v_0 \wedge v_2 \notin v_1) \vee (v_2 \notin v_0 \wedge v_2 \in v_1)\right].$$

Let $g_3(i_0, i_1)$ be a Gödel number for the Σ_1 formula

$$\psi(x, p_\alpha^{n+1}) \equiv \exists v_0, v_1, v_2 \left[\varphi_{i_0}^{(3)}(v_0, (x)_0, p_\alpha^{n+1}) \wedge \varphi_{i_1}^{(3)}(v_1, (x)_1, p_\alpha^{n+1}) \wedge \right. \\ \left. (v_2 \in v_0 \wedge v_2 \notin v_1) \vee (v_2 \notin v_0 \wedge v_2 \in v_1)\right].$$

Then (i_0, x_0) and (i_1, x_1) represent different sets if and only if

$$(g_3(i_0, i_1), (x_0, x_1)) \in A_\alpha^{n+1},$$

which in turn holds if and only if

$$h_X(\pi(g_3(i_0, i_1)), h_X(\pi(x_0), \pi(x_1))) \in M.$$

Since g_3 is computable, it follows from Lemma 4.22 that it is decidable in $X \oplus M$ whether two numbers are π-n-codes and whether they represent the same set. □

Lemma 4.25. *If $\langle X, M \rangle$ is an effective copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1} \rangle$ via π, n_X, h_X, it computes an ω-copy $\langle Y, N \rangle$ of $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$. Furthermore, the computation is uniform, and h_Y and n_Y can be computed uniformly from $X \oplus M \oplus h_X \oplus n_X$.*

Proof. By Lemmas 4.23 and 4.24, the set

$$U = \{y \in \omega \colon \exists\, u \in J_{\rho_\alpha^n}\ (y \text{ is the } <_\omega\text{-least } \pi\text{-}n\text{-code for } u)\}$$

is recursive in $X \oplus M$.

Let σ be the mapping

$$\sigma : u \in J_{\rho_\alpha^n} \mapsto \text{ the unique } \pi\text{-}n\text{-code of } u \text{ in } U\ ,$$

and put

$$Y = \{\langle \sigma(x), \sigma(y)\rangle \colon x \in y \in J_{\rho_\alpha^n}\}, \quad N = \{\sigma(x) \colon x \in A_\alpha^n\}.$$

Then $\langle Y, N\rangle$ is clearly an ω-copy of $\langle J_{\rho_\alpha^n}, A_\alpha^n\rangle$. To show that it is recursive in $X \oplus M$, we note that for $u, w \in J_{\rho_\alpha^n}$, if (i, x) is an n-code for u and (j, y) is an n-code for w,

(4.6)

$$u \in w \;\Leftrightarrow\; \langle J_{\rho_\alpha^n}, A_\alpha^n\rangle \models \exists v_0, v_1\, (\varphi_i(v_0, x, p_\alpha^{n+1}) \wedge \varphi_j(v_1, y, p_\alpha^{n+1}) \wedge v_0 \in v_1).$$

Moreover,

$$u \in A_\alpha^n \;\Leftrightarrow\; \langle J_{\rho_\alpha^n}, A_\alpha^n\rangle \models \exists v_0\, (\varphi_i(v_0, x, p_\alpha^{n+1}) \wedge v_0 \in A_\alpha^n). \tag{4.7}$$

There are recursive functions g_4, g_5 that output Gödel numbers for Σ_1 formulas equivalent to the ones in (4.6) and (4.7), respectively. Given two numbers $a, b \in U$, we can use Lemma 4.22 to find (i, a_0) and (j, b_0) such that $a = h_X(\pi(i), \pi(a_0))$, $b = h_X(\pi(j), \pi(b_0))$. Then

$$aE_Y b \;\Leftrightarrow\; h_X(\pi(g_4(i,j)), h_X(a_0, b_0)) \in M.$$

Likewise,

$$a \in N \;\Leftrightarrow\; h_X(\pi(g_5(i)), \pi(a_0)) \in M.$$

To see that the functions h_Y and n_Y are uniformly recursive in $X \oplus M \oplus h_x \oplus n_X$ note that we can

(i) given $i \in \omega$, effectively compute the Gödel number of a Σ_1 formula that is satisfied by u if and only if u *is* the natural number i,

(ii) given n-codes (i, x), (j, y) for elements u, w in $J_{\rho_\alpha^n}$, compute a Gödel number for a Σ_1 formula whose only solution is (u, w).

□

By iterating the procedure described above, we obtain the following.

Corollary 4.26. *If $\langle X, M\rangle$ is an effective copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1}\rangle$, then it computes ω-copies of*

$$\langle J_{\rho_\alpha^n}, A_\alpha^n\rangle, \langle J_{\rho_\alpha^{n-1}}, A_\alpha^{n-1}\rangle, \ldots, \text{ and } \langle J_{\rho_\alpha^0}, A_\alpha^0\rangle = \langle J_\alpha, \varnothing\rangle = J_\alpha.$$

If a copy is not effective, we can use Lemma 4.19 to decode the predecessor J-structures.

Corollary 4.27. *If $\langle X, M\rangle$ is an ω-copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1}\rangle$, then $(X \oplus M)^{(2)}$ computes ω-copies of*

$$\langle J_{\rho_\alpha^n}, A_\alpha^n\rangle, \langle J_{\rho_\alpha^{n-1}}, A_\alpha^{n-1}\rangle, \dots, \text{ and } \langle J_{\rho_\alpha^0}, A_\alpha^0\rangle = \langle J_\alpha, \varnothing\rangle = J_\alpha.$$

Once we have an ω-copy of J_α, we can use it to compute ω-copies of all J-structures "below" it. We introduce the following notation.

Definition 4.28. Given a structure $\langle F_X, E_X\rangle$ induced by $X \subseteq \omega$ and $z \in \omega$, we define the *segment* $\langle F_X, E_X\rangle\lceil_z$ given by z, as

$$F_X\lceil_z = \{x \in F_X : x E_X z\} \text{ and } E_X\lceil_z = E_X\lceil_{F_X\lceil_z}.$$

In particular, $\langle F_X, E_X\rangle\lceil_z = \varnothing$ if $z \notin F_X$.

If there is no danger of confusing it with the usual initial segment notation for reals, we will abbreviate $\langle F_X, E_X\rangle\lceil_z$ by $X\lceil_z$.

Lemma 4.29. *If X is an ω-copy of J_α, then X computes an ω-copy of $\langle J_{\rho_\beta^n}, A_\beta^n\rangle$, for all $n \in \omega$, $\beta < \alpha$.*

Proof. Both $J_{\rho_\beta^n}$ and A_β^n are elements of J_α. Let π be the isomorphism between J_α and X, and let $x_\beta, a_\beta^n \in F_X$ be such that

$$x_\beta = \pi(J_{\rho_\beta^n}), \quad a_\beta^n = \pi(A_\beta^n).$$

Then $\langle X\lceil_{x_\beta}, \mathrm{Set}_X(a_\beta^n)\rangle$ is an ω-copy of $\langle J_{\rho_\beta^n}, A_\beta^n\rangle$, clearly recursive in X. $\square$

A similar argument yields an analogous fact for the S-operator.

Lemma 4.30. *If X is an ω-copy of J_α, then X computes an ω-copy of $S^{(n)}(J_\beta)$, for all $n \in \omega$, $\beta < \alpha$.*

4.4. Defining ω-copies.

In the previous section we saw how to effectively extract information from ω-copies of J-structures. Next, we describe how ω-copies of new J-structures can be defined from ω-copies of given J-structures.

The J-hierarchy has two types of operations that we need to capture: Defining new sets using the S-operator, and taking projecta and defining standard codes. We will analyze both operations from an arithmetic perspective.

An arithmetic analogue of the S-operator. The S-operator is defined by application of a finite number of explicit functions. This makes it possible to devise an arithmetic analogue, we which denote by $\overline{S}$, and which is the subject of the following lemma.

Lemma 4.31. *There exists an arithmetic function $\overline{S}(X) = Y$ such that, if X is an ω-copy of a transitive set U, $\overline{S}(X)$ is an ω-copy of the transitive closure of $S(U)$. Further, X is coded into a reserved column of ω, that is,*

$$\langle x, y\rangle \in X \Leftrightarrow \langle 2^x, 2^y\rangle \in \overline{S}(X),$$

and 3 represents the element $\{F_X\}$ in $\overline{S}(X)$.

Proof. The elements of $S(U)$ are obtained by single applications of the functions $F_0, \dots, F_8$. Thus each element of $S(U)$ is the denotation of a term consisting of one of the functions and finitely many elements of $U \cup \{U\}$. Recursively in X, we can define an ω-copy of these terms. Membership of the set denoted by one term in the set denoted by another term or equality between the sets denoted by terms is arithmetic in X, since these are defined by quantification over X. The same applies for elements of the transitive closure of the thus coded structure. The additional uniformity condition on the coding of X does not change the calculation. □

We can subject the $\overline{S}$-operator to an analysis similar to that of the jump operator by Enderton and Putnam [10].

Lemma 4.32. *(i) If $A \in 2^\omega$ is an arithmetic singleton, so is $\overline{S}(A)$. Furthermore, the arithmetic complexity of the formula for which $\overline{S}(A)$ is the unique solution is the maximum of the complexity of the formula for A and the complexity of the formula defining $\overline{S}$.*

(ii) There is an arithmetic predicate $Q(n, X, Y)$ such that

$$Q(n, X, Y) \;\Leftrightarrow\; Y = \overline{S}^{(n)}(X).$$

Proof. To prove (i), note that if A is the unique solution to $P(X)$, then $\overline{S}(A)$ is the unique solution to

$$P(X_{[2]}) \text{ and } X = \overline{S}(X_{[2]}),$$

where $X_{[2]} = \{a \colon 2^a \in X\}$.

Claim (ii) follows similarly using the fact that for each k, there is a universal Π^0_k predicate. □

Lemma 4.33. *If Z is such that $Z \geq_{\mathrm{T}} \overline{S}^{(n)}(X)$ for all n, then $\bigoplus_n \overline{S}^{(n)}(X)$ is uniformly arithmetically definable from Z.*

Proof. Define the predicate $\overline{Q}(n, e)$ as

$$\overline{Q}(n, e) \;:\Leftrightarrow\; \Phi_e^Z \text{ is total and } Q(n, X, \Phi_e^Z).$$

To decide whether $a \in \overline{S}^{(n)}(X)$, find, arithmetically in Z, the least e such that $\overline{Q}(n, e)$ and compute $\Phi_e^Z(a)$. □

Corollary 4.34. *If X is an ω-copy of J_α and $Z \geq_{\mathrm{T}} \overline{S}^{(n)}(X)$ for all n, then Z uniformly arithmetically defines an ω-copy of $J_{\alpha+1}$.*

Proof. We can use $\bigoplus_n \overline{S}^{(n)}(X)$ to define a copy of $J_{\alpha+1}$ by 'stacking' the elements of $\overline{S}^{(n+1)}(X)$ coded with base 3 and higher at the next 'available' prime column. Essentially this means that instead of moving $\overline{S}^{(n)}(X)$ into the column given by powers of 2, we leave it unchanged and add new elements for $\overline{S}^{(n+1)}(X)$ starting at the smallest available prime column. □

An arithmetic version of the standard code. To define an arithmetic copy the Σ_n-standard code for J_α, we can simply interpret the set theoretic definitions as formulas of arithmetic. More precisely, suppose P is a definable predicate over a J-structure $\langle J_{\rho_\alpha^n}, A_\alpha^n\rangle$, and $\langle X, M\rangle$ is an ω-copy via π. Since the structure $\langle F_X, E_X, M\rangle$ is isomorphic to $\langle J_{\rho_\alpha^n}, \in, A_\alpha^n\rangle$, we can use the same formula that defines P over $\langle J_{\rho_\alpha^n}, A_\alpha^n\rangle$ and obtain a definition of $\pi[P]$ arithmetic in $\langle X, M\rangle$. The problem, however, is that a bounded quantifier in set theory will not necessarily correspond to a bounded quantifier in arithmetic. This means the transfer of complexities between the Lévy-hierarchy and the arithmetical hierarchy may not result in uniform bounds.

However, we will use only a fixed, finite number of set-theoretic definitions. Most importantly, we use the uniform definability of the satisfaction relation $\models$ over transitive, rud closed structures.

Proposition 4.35 (Jensen [22], Corollary 1.13)**.** *For $n \geq 1$, the satisfaction relation $\models^{\Sigma_n}_{\langle M,A\rangle}$ is uniformly $\Sigma_n(\langle M, A\rangle)$ for transitive, rud closed structures $\langle M, A\rangle)$.*

Corollary 4.36. *Suppose $\langle X, M\rangle$ is an ω-copy of $\langle J_{\rho_\alpha^n}, A_\alpha^n\rangle$. Then there exists an ω-copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1}\rangle$ uniformly arithmetically definable in $\langle X, M\rangle$.*

4.5. **Recognizing J-structures.** Our goal is to show that there exists a recursive function G such that, for each n, no element of the sequence of canonical copies of J-structures with projectum equal to one in L_{β_n} can be $G(n)$-random with respect to a continuous measure. In the proof of this result (Theorem 4.50), we need to consider the initial segment of ω-copies computable in (some fixed jump of) μ.

The problem is that we cannot arithmetically define the set of ω-copies of structures J_α. We can, however, define a set of "pseudocopies", subsets of ω that behave in most respects like ω-copies of actual J_α, but that may code structures that are not well-founded.

By comparing the structures coded by these pseudocopies, we can also linearly order a subset of the latter (up to isomorphism), depending on whether a coded structure embeds into another. This ordering will be developed in Section 4.6.

Definition 4.37. A set $X \subseteq \omega$ is a *pseudocopy* if the following hold.

(1) The relation E_X is non-empty and extensional.
(2) The structure $\langle F_X, E_X\rangle$ is rud-closed.
(3) The structure $\langle F_X, E_X\rangle$ satisfies $\varphi_{V=J}$.
(4) X's version of ω is isomorphic to ω; that is, X codes an ω-model.

To formalize these properties in arithmetic, we can take, as before, any formula in the language of set theory and interpret it over the structure $\langle F_X, E_X\rangle$. This way we can define relations over F_X intended to represent the corresponding set-theoretic relation. Extensionality can be formalized by a $\Pi^0_2(X)$ formula:

$$\forall x, y(\forall z(z E_X x \leftrightarrow z E_X y) \;\rightarrow\; x = y).$$

Being rud-closed is an arithmetic property relative to X.

By Mostowski's Collapsing Theorem, if X satisfies (1) and E_X is well-founded, then $\langle F_X, E_X\rangle$ is isomorphic to a transitive set structure $\langle S, \in\rangle$, and by (2) S will be rud-closed.

Property (3) is clearly arithmetic since it is defined by a single formula.

Finally, for (4), we can define ω using the usual Σ_0 set theoretic formula (the least infinite ordinal). In transitive, rud-closed sets, $\varphi_\omega(x)$ holds if and only if $x = \omega$. Interpreting φ_ω over $\langle F_X, E_X\rangle$, we obtain an arithmetic in X

property. We require a pseudocopy $\langle F_X, E_X \rangle$ to satisfy

$$\exists x\ \varphi_\omega(x).$$

This x will be unique and define ω with respect to $\langle F_X, E_X \rangle$. Let us denote this unique number by ω_X.

Given ω_X, we can also recover the mapping $i \mapsto n_X(i)$ as in Lemma 4.17. As the definition of ω_X is uniform, we obtain that $i \mapsto n_X(i)$ is uniformly arithmetic in X. (4) holds exactly when this map from ω to ω_X is a surjection, which is again uniformly arithmetic relative to X.

Lemma 4.38. *There exists an arithmetic formula $\varphi_{\mathrm{PC}}(X)$ such that if $\varphi_{\mathrm{PC}}(X)$ holds for a real X, then X is a pseudocopy. Moreover, if $\langle F_X, E_X \rangle$ is well-founded, then it is an ω-copy of a countable J_β, $\beta > 1$.*

4.6. **Comparing pseudocopies.** If two pseudocopies X and Y define well-founded structures, they are ω-copies of sets J_α and J_β, respectively. Since $\alpha < \beta$ implies $J_\alpha \in J_\beta$, it follows that one structure must embed into the other as an initial segment.

We want to find an arithmetic formula that compares two pseudocopies in this respect. The problem is that the isomorphism relation between countable structures need not be arithmetic. In our case, however, we can make use of the special set-theoretic structure present in the pseudocopies, by comparing the subsets of the cardinals present.

The complexity of the arithmetic operations involved in these comparisons will depend on the number of cardinals present in a pseudocopy.

Let us introduce the following notation. Recall that β_N denotes the least ordinal such that $L_{\beta_N} \models \mathsf{ZF}^-_N$. For any ordinal α, let

$$P_\alpha = \max\{n \colon \mathcal{P}^{(n)}(\omega) \text{ exists in } J_\alpha\}, \tag{4.8}$$

if this maximum exists. We first note that for all $\alpha < \beta_N$, $P_\alpha \leq N$. This is because, if P_α were greater than or equal to $n+1$ and β were the $(n+1)$st cardinal in L_α, then L_β would satisfy ZF^-_N, hence $\alpha > \beta_N$. Hence P_α is defined and uniformly bounded by N for all $\alpha < \beta_N$.

Using the predicate φ_ω, we can formalize the (non-)existence of power sets of ω for pseudocopies. For any $k \in \omega$, there exists a formula defining the predicate

$$y = \mathcal{P}^{(k)}(\omega).$$

Definition 4.39. A pseudocopy X is an *n-pseudocopy* if it satisfies the uniformly arithmetic in X predicate

$$\exists y(y = \mathcal{P}^{(n)}(\omega)) \;\wedge\; \forall z(z \neq \mathcal{P}^{(n+1)}(\omega)).$$

We now use the fact that pseudocopies are ω-models. Using the power sets of ω in each pseudocopy, we can check whether two pseudocopies have the same reals, sets of reals, etc.

First, we can check whether every real in X has an analogue in Y:

$$\forall u\,(X \models u \subseteq \omega \;\to\; \exists v(Y \models v \subseteq \omega \;\wedge\; \forall i(n_X(i) E_X u \;\leftrightarrow\; n_Y(i) E_Y v))).$$

By extensionality, such a v, if it exists, is unique. We can therefore define the mapping $f_0^{X,Y}(u) = v$ which maps the representation of a real in $\langle F_X, E_X\rangle$ to its representation in $\langle F_Y, E_Y\rangle$. We can similarly check whether every real in Y has an analogue in X. This gives rise to a function $f_0^{Y,X}$. Let $\varphi^{(0)}_{\text{comp}}(X,Y)$ be the arithmetic formula asserting that X and Y code the same subsets of ω.

We can continue this comparison through the iterates of the power set of ω. This will yield arithmetic formulas $\varphi^{(n)}_{\text{comp}}(X,Y)$ with the following property:

> If X and Y are pseudocopies in which $\mathcal{P}^{(n)}(\omega)$ exists, then $\varphi^{(n)}_{\text{comp}}(X,Y)$ holds if and only if X and Y have (representations of) the same subsets of $\mathcal{P}^{(i)}(\omega)$, for all $0 \leq i \leq n$.

Given two n-pseudocopies, the above formulas allow for an arithmetic definition of isomorphic pseudocopies.

Lemma 4.40. *For given n and for any two n-pseudocopies X, Y that code well-founded structures $\langle F_X, E_X\rangle$ and $\langle F_Y, E_Y\rangle$, respectively, if $\varphi^{(n)}_{\text{comp}}(X,Y)$, then X and Y code the same J_α.*

Proof. Assume for a contradiction X and Y are not isomorphic. Since they are well-founded pseudocopies, there must exist countable α, β such that $\langle F_X, E_X\rangle \cong (J_\alpha, \in)$ and $\langle F_Y, E_Y\rangle \cong (J_\beta, \in)$. Without loss of generality, $\alpha < \beta$. Since $\langle F_X, E_X\rangle$ and $\langle F_Y, E_Y\rangle$ code the same subsets of $\mathcal{P}^{(n)}(\omega)$ no new subset of $\mathcal{P}^{(n)}(\omega)$ is constructed between α and β. But this implies $\mathcal{P}^{(n+1)}(\omega)$ exists at $\alpha+1$, which is an immediate contradiction if $\beta = \alpha+1$. If $\beta > \alpha + 1$, since $\mathcal{P}^{(n+1)}(\omega)$ does not exist in J_β, a new subset of $\mathcal{P}^{(n)}(\omega)$ must be constructed between $\alpha+1$ and β, contradiction. □

We will consider the comparison between ill-founded structures later.

Corollary 4.41. *For given n, if X and Y are well-founded n-pseudocopies and $\varphi^{(n)}_{\text{comp}}(X,Y)$, then there exists an arithmetically in (X,Y) definable function which maps the structure coded by X isomorphically onto the structure coded by Y.*

Proof. Under the given hypothesis, there exists an α such that both X and Y are isomorphic to J_α. By Proposition 4.4, there is a definable map from $\omega\alpha$ onto J_α. Moreover, there is a definable map from subsets of the greatest cardinal in J_α onto $\omega\alpha$. Finally, there is a definable bijection from the greatest cardinal, which is some $\omega_k^{J_\alpha}$, to subsets of $\mathcal{P}^{(k)}(\omega)$. This gives us a definable isomorphism: We map an element in X to its least pre-image in $\omega\alpha$. This then gets mapped to the subset of $\mathcal{P}^{(k)}(\omega)$ to which it corresponds in X. Then we use our comparison formula between subsets of $\mathcal{P}^{(k)}(\omega)$ to map it to its counterpart in Y. Finally, in Y we map the subset of $\mathcal{P}^{(k)}(\omega)$ obtained this way to an ordinal and consequently to an element of Y. □

We can use the transfer function of Corollary 4.41 to translate also between copies of $S^{(n)}(J_\alpha)$.

Corollary 4.42. *For every N, there exists a number d_N, which can be computed uniformly from N, such that the following holds. Suppose X is an ω-copy of some J_α with $P_\alpha \leq N$. Suppose further that Z is an ω-copy of $S^{(n)}(J_\alpha)$, for some $n \in \omega$. Then $\overline{S}^{(n)}(X)$ is recursive in $(X \oplus Z)^{(d_N)}$.*

Proof. As Z computes an ω-copy of J_α, Corollary 4.41 implies that some jump of $Z \oplus X$ computes the isomorphism between the two ω-copies of J_α. Now apply Lemma 4.32. □

For fixed $N \in \omega$, let

$$\mathcal{PC}_N = \{X \colon X \text{ is an } n\text{-pseudocopy for some } n \leq N\}.$$

This is an arithmetic set of reals.

Restricted to $\mathcal{PC}_N$, the following relation, denoted by $\sim_N$ is arithmetically definable:

$$\begin{aligned} X \sim_N Y :\Leftrightarrow\ & X, Y \text{ are 0-pseudocopies and } \varphi^{(0)}_{\text{comp}}(X,Y) \vee \\ & X, Y \text{ are 1-pseudocopies and } \varphi^{(1)}_{\text{comp}}(X,Y) \vee \\ & \quad \vdots \\ & X, Y \text{ are } N\text{-pseudocopies and } \varphi^{(N)}_{\text{comp}}(X,Y) \end{aligned}$$

Working inside $\mathcal{PC}_N$, we can also use φ_{comp} to arithmetically define a preorder $\prec$ on pseudocopies. The idea is that $X \prec Y$ if X embeds its structure into Y. For this purpose, we have to identify the "internal" J-hierarchy of a pseudocopy.

By Proposition 4.2, for any β, the sequence of J_α ($\alpha < \beta$) is uniformly Σ_1-definable over J_β. Let φ be the defining formula.

For $z \in F_X$, we define

$$J^X = \{z \colon X \models \varphi(z)\},$$

the J-structure inside X (the *internal J-structure of X*) and, given $z \in J^X$, write J^X_z for $X\lceil_z$.

Lemma 4.43. *Suppose $X \in \mathcal{PC}_N$ and $z \in J^X$. Then J^X_z has no non-trivial automorphism. Further, if $J^X_{z_1}$ is isomorphic to $J^X_{z_2}$, then $z_1 = z_2$.*

Proof. Let π be an automorphism of J^X_z. Since X codes an ω-model, J^X_z is an ω-model. π fixes the ω of J^X_z and also fixes every natural number of J^X_z. By induction π must fix every set that is obtained by a finite number of power set operations to ω. There is an internally definable injection from J^X_z to sets obtained by a finite number of power set operations to ω. Hence π must fix all of J^X_z. (See proof of Corollary 4.41.)

The second claim follows by the same reasoning. □

If X is well-founded, J^X must be linearly ordered by E_X. So if it is not (an arithmetic property relative to X), we can exclude X as ill-founded right away. From now on suppose J^X is always linearly ordered.

Definition 4.44. We define

$$X \prec_N Y \quad :\Leftrightarrow X, Y \in \mathcal{PC}_N \ \wedge\ \exists z\,(z \in J^Y \ \wedge\ J^Y_z \in \mathcal{PC}_N \ \wedge\ X \sim_N J^Y_z)).$$

This is an arithmetic property of the pair (X, Y). We let $X \preceq_N Y$ if $X \prec_N Y$ or $X \sim_N Y$. $\preceq_N$ is reflexive and transitive. Hence $\preceq_N$ defines a partial order on $\mathcal{PC}_N$.

If both X and Y are well-founded pseudocopies in $\mathcal{PC}_N$, we have either $X \prec_N Y$ or $X \sim_N Y$ or $Y \prec_N X$, that is, "true" pseudocopies (i.e., those that code a J_α) are linearly ordered by $\preceq_N$ (up to isomorphism). Hence comparability can only fail if (at least) one of the pseudocopies is not well-founded.

Provided with a countable subset of $\mathcal{PC}_N$, such as all the elements of $\mathcal{PC}_N$ recursive in a real Z, we will want to arithmetically define a subset that is linearly ordered by $\preceq_N$ by excluding some ill-founded pseudocopies.

Definition 4.45. Given a real Z, let $\mathcal{PC}_N(Z)$ be the set of pseudocopies computable in Z.

Lemma 4.46. *For every natural number N there is an arithmetic predicate such that for every real Z, the predicate defines a set of reals $\mathcal{PC}^*_N(Z) \subseteq \mathcal{PC}_N(Z)$ with the following properties:*

(1) *For every $X \in \mathcal{PC}^*_N(Z)$ and $x \in F_X$, if $J^X(x)$, then $J^X_x \in \mathcal{PC}^*_N(Z)$,*
(2) *$\preceq_N$ is a total preorder on $\mathcal{PC}^*_N(Z)$, i.e., $\mathcal{PC}^*_N(Z)/\sim_N$ is linearly ordered,*
(3) *If $X \in \mathcal{PC}_N(Z)$ is well-founded, then $X \in \mathcal{PC}^*_N(Z)$.*

Proof. For property (1), suppose $X \in \mathcal{PC}_N(Z)$ and $z \in J^X$. We check that $J^X_z \in \mathcal{PC}_N(Z)$. J^X_z is clearly recursive in Z. Since X codes an ω-model, properties (1)-(4) of Definition 4.37, which are satisfied by J^X_z within X, are true of J^X_z.

For property (2), we investigate $\preceq_N$-incomparability. Suppose X, Y are $\preceq_N$-incomparable. We can use the $\sim_N$-relation to see if two J-segments of X and Y align: Consider the predicate

$$J^X(x) \wedge J^Y(y) \wedge J^X_x \sim_N J^Y_y.$$

It yields an arithmetic partial function from F_X to F_Y. It is single-valued by Lemma 4.43. Denote the domain of this function by D_X and the range by R_Y. Both D_X and R_Y are linearly ordered.

We consider the set of ordinals in a pseudocopy:

$$\mathrm{Ord}_X = \{z\colon\ z \text{ is an ordinal in } X\},$$
$$\mathrm{Ord}_Y = \{z\colon\ z \text{ is an ordinal in } Y\}.$$

$\mathrm{Ord}_X, \mathrm{Ord}_Y$ are closed downward under E_X, (E_Y, respectively), and linearly ordered by E_X (E_Y). If one structure is ill-founded, it must exhibit an instance of ill-foundedness among its ordinals, since an infinite descending $\in$-chain in X would yield an infinite descending chain in the J^X-hierarchy, which would correspond to an infinite descending chain in the ordinals in X.

Let

$$\mathrm{Ord}(D_X) = \bigcup\{\mathrm{Ord}_X \cap J^X_z : z \in D_X\}, \quad \mathrm{Ord}(R_Y) = \bigcup\{\mathrm{Ord}_Y \cap J^Y_z : z \in R_Y\}.$$

Both $\mathrm{Ord}(D_X)$ and $\mathrm{Ord}(R_Y)$ are initial segments of Ord_X and Ord_Y respectively, because the ordinals of any element of the J-hierarchy are an initial segment of the ordinals.

Now we apply the incomparability of X and Y to show that there must be at least one instance of ill-foundedness in $\mathrm{Ord}(D_X)$ or $\mathrm{Ord}(R_Y)$.

Case 1: $\mathrm{Ord}(D_X)$ *is cofinal in* Ord_X*,* $\mathrm{Ord}(R_Y)$ *is cofinal in* Ord_Y*.*
This means, by the definition of the function for which D_X and R_Y are domain and range, respectively, the complete internal J-hierarchies of X and Y, respectively, are pairwise isomorphic. Furthermore, these isomorphisms are compatible by Lemma 4.43. Their union hence exhibits an isomorphism between the structure coded by X and the structure coded by Y, which would imply $X \sim_N Y$.

Case 2: $\mathrm{Ord}(D_X)$ *is cofinal in* Ord_X*,* $\mathrm{Ord}(R_Y)$ *is bounded in* Ord_Y*.*
Since $X \not\preceq_N J_y$ for any y, Y must omit $\bigcup_{z \in R_Y} J_z$ and hence is ill-founded. The case when $\mathrm{Ord}(D_X)$ is bounded and $\mathrm{Ord}(R_Y)$ is cofinal is analogous.

Case 3: *Both* $\mathrm{Ord}(D_X), \mathrm{Ord}(R_Y)$ *are bounded in* $\mathrm{Ord}_X, \mathrm{Ord}_Y$*, respectively.*
In this case, $\mathrm{Ord}(D_X)$ and $\mathrm{Ord}(R_Y)$ are cuts in Ord_X and Ord_Y, respectively. If these cuts were principal in both structures, it would contradict the definition of D_X and R_Y by adding a new element to each set. In the limit case, reason as in Case 1: the union of the J_x^X, $x \in D_x$, as evaluated in X, maps to the union of the J_y^Y, $y \in R_Y$, as evaluated in Y. In the successor case, given an isomorphism between J_x^X and J_y^Y, because X and Y code ω-models, there also exists an isomorphism between $\overline{S}(J_x^X)$, as evaluated in X, and $\overline{S}(J_y^Y)$, as evaluated in Y.

Therefore, at least one of the two cuts is not principal, thereby exhibiting an instance of non-wellfoundedness.

We thus obtain the desired linearization of $\preceq_N$. However, its definition involves quantification over *all* pseudocopies in $\mathcal{PC}_N$, and is therefore, if unrestricted, not arithmetic. We obtain the arithmetic set $\mathcal{PC}^*_N(Z)$ by considering all incomparable pairs in $\mathcal{PC}_N(Z)$ and discarding all elements of $\mathcal{PC}_N(Z)$ that are shown to be ill-founded by the above analysis. Since all pairs are being considered, $\mathcal{PC}^*_N/\cong_N$ is linearly ordered by $\prec_N$.

To see that this also ensures property (3) of the lemma, note that any element removed from $\mathcal{PC}_N(Z)$ in this process is ill-founded. □

Taking limits of ω-copies. We can use the ordering $preceq_N$ to construct limits of ω-copies. This will be needed in the proof of Theorem 4.50.

Lemma 4.47. *For every N, there exists a number d_N, which can be computed uniformly from N, such that the following holds. Suppose $\overline{X} = \{X_i \colon i \in \omega\}$ is a family of well-founded pseudocopies from $\mathcal{PC}_N$, in other words, each each X_i codes a countable J_{α_i} in which there are at most N uncountable cardinals. Let γ be the supremum of the α_i. Then there exists an ω-copy of J_γ recursive in $\overline{X}^{(d)}$.*

Proof. Using the $\preceq_N$-predicate, we can arithmetically define a function $f : \omega \to \omega$ such that for all i, j,

$$f(i) \leq f(j) \ \Leftrightarrow \ X_i \preceq_N X_j.$$

This gives us a directed system of copies. We define a copy of J_γ as a copy of the union of this directed system.

Let $Y_0 = X_{f(0)}$. Initialize by putting

$$U_0 = \{\langle 2^{x+1}, 2^{y+1}\rangle \colon \langle x, y\rangle \in Y_0\}.$$

Suppose now we have defined $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_l$ with the property that

$$F_{U_i} \subseteq \bigcup_{k=0}^{i} \{p_i^m \colon m \in \omega\},$$

where p_i is the i-th prime number. If $X_{f(l+1)} \sim_N X_{f(l)}$, put $U_{l+1} = U_l$. Otherwise, pick z such that $X_{f(l+1)}\lceil_z$ is isomorphic to U_l, and let π_{l+1} be the isomorphism between $X_{f(l+1)}\lceil_z$ and U_l. Given x, y such that $x \, E_{X_{f(l+1)}} \, y$, define $E_{U_{l+1}}$ as follows:

- If both $x \, E_{X_{f(l+1)}} \, z$ and $y \, E_{X_{f(l+1)}} \, z$, add $\langle \pi_{l+1}(x), \pi_{l+1}(y)\rangle$ to U_{l+1}.
- If $x \, E_{X_{f(l+1)}} \, z$ but not $y \, E_{X_{f(l+1)}} \, z$, add $\langle \pi_{l+1}(x), p_{l+1}^{y+1}\rangle$ to U_{l+1}.
- If neither $x \, E_{X_{f(l+1)}} \, z$ nor $y \, E_{X_{f(l+1)}} \, z$, add $\langle p_{l+1}^{x+1}, p_{l+1}^{y+1}\rangle$ to U_{l+1}.

Putting

$$U = \bigcup_{l \in \omega} U_l$$

yields an ω-copy of J_γ arithmetic in $\overline{X}$. □

4.7. **Canonical copies are not random for continuous measures.** We now want to use the framework of ω-copies to show that for any $\alpha < \beta_N$, the canonical copy of a standard J-structure $\langle J_{\rho_\alpha^k}, A_\alpha^k\rangle$ cannot be K-random for a continuous measure, with K sufficiently large.

The argument rests mostly on various applications of the stair trainer technique introduced in Propositions 2.16 and 2.17, adapted to the notions of codings of countable J-structures developed in the previous sections. For convenience, we briefly review the core concepts.

ω-copy: A coding of a countable set-theoretic structure $\langle S, A\rangle$, $A \subseteq S$, as a subset of ω; see Definitions 4.13 and 4.14.

Canonical copy: The copy of a J-structure $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$ with projectum $\rho^n_\alpha = 1$ by means of a canonical bijection $V_\omega \leftrightarrow \omega$; see Definition 4.15.

Effective copy: An ω-copy of a J-structure $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$ from which the internal fine structure hierarchy can effectively be recovered; see Definition 4.20 and Corollary 4.26. A canonical copy is always effective.

Pseudocopy: An ω-copy of an ω-model of a rud-closed set satisfying $\varphi_{V=J}$. The set of pseudocopies is arithmetically definable. A pseudocopy may be ill-founded. If it is well-founded, it codes some countable level J_α of the J-hierarchy; see Definition 4.37 and Lemma 4.38. By comparing their internal J-hierarchies, a subset of pseudocopies can be linearly ordered (up to isomorphism). This linear ordering is arithmetic, too; see Lemma 4.46.

We also fix some notation for the rest of this section. Given $N \in \omega$, we fix $c \in \omega$ to be a sufficiently large number. It will be greater than the complexity of all arithmetic definitions (N-pseudocopies, comparison of pseudocopies and S-operators, linearization) introduced in the previous sections. In particular, Corollary 4.41 yields that, if X and Y are well-founded ω-copies of a J_α where $\alpha < \beta_N$, then the isomorphism between the two coded structures is recursive in $(X \oplus Y)^{(c)}$. It will also be greater than the numbers d_N from Corollary 4.42 and Lemma 4.47. After proving a couple of auxiliary results (Lemmas 4.48 and 4.49), we will also assume c to be greater than the constants appearing in these lemmas.

We give a first application of the stair trainer technique (as used in the proofs of Propositions 2.16 and 2.17) in the context of ω-copies and pseudocopies.

Lemma 4.48. *There exist numbers $d, e \in \omega$ such that the following holds. Suppose μ is a continuous measure and X is an ω-copy of J_α recursive in $\mu^{(m)}$, for some $m \in \omega$. Suppose further that R is $(m+e)$-random with respect to μ and computes an ω-copy of $J_{\alpha+1}$. Then there exists an ω-copy of $J_{\alpha+1}$ recursive in $\mu^{(m+d)}$.*

Proof. By Lemma 4.30, R computes an ω-copy of $S^{(n)}(J_\alpha)$, for all $n \in \omega$. By Corollary 4.42, $\overline{S}^{(n)}(X)$ is recursive in $(R \oplus \mu)^{(m+d_N)}$. Lemma 4.31 on the other hand implies that $\overline{S}(X)$ is recursive in $\mu^{(m+a)}$, for some fixed a. We may assume that R is $(m+d_N+a+1)$-random for μ. By Proposition 2.12 and Lemma 2.14, $\overline{S}(X)$ is recursive in $\mu^{(m+d_N)}$. We can inductively use this line of reasoning and obtain that for each $n \in \omega$, $\overline{S}^{(n)}(X)$ is recursive in $\mu^{(m+d_N)}$. Now apply Corollary 4.34. □

The lemma shows that with the help of a sufficiently random real that computes an ω-copy of the next level of the J-hierarchy, μ can reach a copy of this level arithmetically, too. Combined with Lemma 4.47, this will be the key ingredient in proving that canonical copies of standard codes cannot be random with respect to a continuous measure.

The next lemma establishes a similar fact for the standard J-structures $\langle J_{\rho_\delta^n}, A_\delta^n \rangle$ over a given J_δ.

Lemma 4.49. *There exist numbers $d, e \in \omega$ such that the following holds. Suppose μ is a continuous measure and X is an ω-copy of J_δ. Further suppose X is recursive in $\mu^{(m)}$. Finally, suppose that R is $(m+e)$-random with respect to μ and n is such that R computes an ω-copy of $\langle J_{\rho_\delta^n}, A_\delta^n \rangle$. Then there exists an ω-copy $\langle X_\delta^n, M_\delta^n \rangle$ of $\langle J_{\rho_\delta^n}, A_\delta^n \rangle$ recursive in $\mu^{(m+d)}$.*

Proof. The proof is similar to that of Lemma 4.48, inductively using Proposition 2.12, Lemma 2.14, Corollary 4.36, and Corollary 4.41. □

From now on, we assume that c is also greater than the respective constants d, e from Lemmas 4.47, 4.48 and 4.49. We define $G(N) = (N+1)(3c+6)$.

Theorem 4.50. *Suppose $N \geq 0$, $\alpha < \beta_N$, and for some $m > 0$, $\rho_\alpha^m = 1$. Then the canonical copy of the standard J-structure $\langle J_{\rho_\alpha^m}, A_\alpha^m \rangle$ is not $G(N)$-random with respect to any continuous measure.*

Proof. We fix R and m_R so that, when R is interpreted as a pair of reals, R is the canonical copy of some standard J-structure $\langle J_{\rho_\alpha^{m_R}}, A_\alpha^{m_R} \rangle$, where $\rho_\alpha^{m_R} = 1$. We assume for the sake of a contradiction that R is $G(N)$-random with respect to a continuous measure μ.

To obtain a contradiction similar to the proofs of Propositions 2.16 and 2.17, we inductively define a hierarchy of indices of (pseudo)copies arithmetic in μ.

Definition 4.51. For each k with $0 \leq k \leq N$, we let

$$S_k = \{e \in \omega \colon \Phi_e^{\mu^{(k(3c+6))}} \text{ is total and } \Phi_e^{\mu^{(k(3c+6))}} \in \mathcal{PC}_N(\mu^{(k(3c+6))}) \text{ and for all } d < e,\ \Phi_d^{\mu^{(k(3c+6))}} \neq \Phi_e^{\mu^{(k(3c+6))}} \}.$$

The relation $\prec_N$ induces an ordering on the indices in each S_k, which will be denoted by $\prec_N$, too. The linearly ordered subsets corresponding to $\mathcal{PC}_N^*$ are given as

$$L_k = \{e \in S_k \colon \Phi_e^{\mu^{(k(3c+6))}} \in \mathcal{PC}_N^*(\mu^{(k(3c+6))})\}.$$

Finally, we let

$$I_k = \{e \in S_k \colon \Phi_e^{\mu^{(k(3c+6))}} \text{ is well-founded}\}.$$

By Lemma 4.46, the sets S_k and L_k are arithmetic in $\mu^{(k(3c+6))}$, for all $k \leq N$. In particular, by choice of c, L_k is recursive in $\mu^{(k(3c+6)+c)}$.

The following lemma shows that I_k is the longest well-founded initial segment of L_k.

Lemma 4.52. *Given $k \leq N$, let I be a well-founded initial segment of L_k. Then, for every $e \in I$, $\Phi_e^{\mu^{(k(3c+6))}}$ is a well-founded pseudocopy.*

Proof of Lemma 4.52. Suppose $\Phi_e^{\mu^{(k(3c+6))}}$, $e \in I$, is an ill-founded pseudocopy. Then it has an ill-founded sequence of ordinals, and hence also an ill-founded internal J-sequence. Since I is an initial segment, the entire internal J-structure must be present in I (via corresponding $\mu^{(k(3c+6))}$-indices); see Lemma 4.46. This contradicts the fact that I is well-founded. □

We will need an additional properties of I_k.

Lemma 4.53. *If J_β is represented in I_k, then $\beta < \alpha$.*

Proof of Lemma 4.53. Suppose J_α is represented in I_k. By Lemma 4.49, there exists an ω-copy $\langle X, M\rangle$ of $\langle J_{\rho_\alpha^{m_R}}, A_\alpha^{m_R}\rangle$ recursive in $\mu^{(k(3c+6)+c)}$. (Recall m_R is such that $\rho_\alpha^{m_R} = 1$ and $R = \langle J_1, A_\alpha^{m_R}\rangle$.) Comparing the canonical encoding of J_1 with X, we obtain that R is recursive in $\mu^{(k(3c+6)+2c)}$, contradicting the randomness of R. □

We continue the proof of Theorem 4.50 and apply Lemma 2.15 to I_k. L_k is recursive in $\mu^{(k(3c+6)+c)}$. Since R is a canonical copy, we can use it to test for any pseudocopy M with an index in L_k whether M embeds into J_α. This can be done recursively in $(\mu^{(k(3c+6))} \oplus R)^{(c)}$ by Lemma 4.46 and the choice of c. By Lemma 4.52, every pseudocopy with an index in I_k will embed into

J_α. Consequently, I_k is recursive in $(\mu \oplus R)^{(k(3c+6)+c)}$. By choice of G, R is at least $(k(3c+6)+c+5)$-random for μ, so Lemma 2.15 implies that I_k is recursive in $\mu^{(k(3c+6)+c+4)}$.

Next we define by recursion a sequence of ordinals $\gamma_0, \ldots, \gamma_K$, where K is at most $N+1$.

- Let $\gamma_0 = \omega$ and $\xi_0 = 1$.
- Given γ_k, we check whether γ_k is a cardinal in each of the structures represented in I_k.
  - If so, we let
    $$\begin{aligned}\gamma_{k+1} = \sup\{\beta \colon \exists e \in I_k\ (&\beta \text{ has cardinality at most } \gamma_k \\ &\text{in the structure represented by } e)\},\\ \xi_{k+1} = \sup\{\beta \colon\ &J_\beta \text{ is represented in } I_k\},\end{aligned}$$
    and we continue the recursion.
  - Otherwise, there exists a $j \leq k$ such that γ_j is not a cardinal inside some structure J_δ represented in I_k. Since the recursion made it to step k, δ is greater than any β such that J_β has a representation in I_{k-1}. We terminate the recursion and let $K = k$.
- If we reach γ_{N+1}, we terminate the recursion.

Lemma 4.54. *For every $0 \leq k \leq K$, J_{ξ_k+1} is represented in I_k.*

Proof of Lemma 4.54. We first prove that J_2 is represented in I_0. The canonical copy of J_1 is recursive. By Lemma 4.48, there exists an ω-copy of J_2 recursive in $\mu^{(c)}$. The canonical copy of J_2, $\langle J_1, A_2\rangle$ is recursive in R. By Lemma 4.49, there exists an ω-copy of $\langle J_1, A_2\rangle$ recursive in $\mu^{(2c)}$. The isomorphism between this copy and the canonical copy of J_1 is recursive in $\mu^{(3c)}$. Therefore, by Lemma 2.13, the canonical copy of $\langle J_1, A_2\rangle$ is recursive in μ. By Corollary 4.26, there exists an ω-copy of J_2 recursive in μ.

Now assume $k > 0$. J_{ξ_k} has a representation recursive in $\mu^{((k-1)(3c+6)+c)}$. If ξ_k is the maximum of the β for which J_β is represented in I_{k-1}, there is an ω-copy of J_{ξ_k} recursive in $\mu^{((k-1)(3c+6))}$. Otherwise, Lemma 4.47 implies that there exists an ω-copy of J_{ξ_k} recursive in $\mu^{((k-1)(3c+6)+c)}$. We can apply Lemma 4.48 to obtain an ω-copy of J_{ξ_k+1} recursive in $\mu^{((k-1)(3c+6)+2c)}$, which implies that it is represented in I_k. □

The lemma implies that for each $i < K$, γ_i appears as a cardinal in some structure represented in I_i.

Case 1: $K < N+1$, that is, the recursion terminates early.

In this case either the projectum ρ_δ of J_δ is 1 or there is an $1 < i < K$ such that ρ_δ is γ_i. This is because for every infinite ordinal less than γ_K, there is a structure represented in I_{K-1} in which this ordinal is in one-to-one correspondence with some γ_i, $i < K$.

Case 1a: $\rho_\delta = 1$. The canonical copy of J_1 is recursive. By Lemma 4.49, there exists an ω-copy of $\langle J_1, A_\delta\rangle$ recursive in $\mu^{(K(3c+6)+c+4+c)} = \mu^{(K(3c+6)+2c+4)}$. By Corollary 4.27, the canonical copy $\langle X, M\rangle$ of $\langle J_1, A_\delta\rangle$ is recursive in $\mu^{(K(3c+6)+2c+4+2)} = \mu^{(K(3c+6)+2c+6)}$. Since $\alpha > \gamma_K$, R computes $\langle X, M\rangle$, by Lemma 4.29. By Lemma 2.13, μ computes $\langle X, M\rangle$. Since $\langle X, M\rangle$ is an effective copy, Corollary 4.26 implies μ computes an ω-copy of J_δ. But this means J_δ is represented in I_0, which contradicts the definition of γ_1.

Case 1b: $\rho_\delta > 1$. Note that by Lemma 4.47, there exists an ω-copy Y of J_{ρ_δ} recursive in $\mu^{((i-1)(3c+6)+2c+4)}$.

Since J_δ is represented in I_K, there exists an ω-copy of J_δ recursive in $\mu^{(K(3c+6))}$. By Lemma 4.49, there exists an ω-copy $\langle X, M\rangle$ of $\langle J_{\rho_\delta}, A_\delta\rangle$ recursive in $\mu^{(K(3c+6)+c)}$.

By comparing the coding of X and Y (using at most c jumps), $\mu^{(K(3c+6)+c+c)} = \mu^{(K(3c+6)+2c)}$ can compute the transfer of M (the coding of A_δ in X) to Y. This gives us an ω-copy $\langle Y, L\rangle$ of $\langle J_{\rho_\delta}, A_\delta\rangle$ recursive in $\mu^{(K(3c+6)+2c)}$.

Since $\alpha > \gamma_K$, R computes, by Lemma 4.29, another ω-copy of $\langle J_{\rho_\delta}, A_\delta\rangle$, say $\langle Y_R, L_R\rangle$. Using at most c jumps, the join of R and $\mu^{((i-1)(3c+6)+2c+4)}$ can compare Y and Y_R and map L_R, the encoding of A_δ in Y_R, to Y. This way we obtain an ω-copy $\langle Y, L_Y\rangle$ of $\langle J_{\rho_\delta}, A_\delta\rangle$ recursive in $(R \oplus \mu^{((i-1)(3c+6)+2c+4)})^{(c)} \equiv_{\mathrm{T}} R \oplus \mu^{((i-1)(3c+6)+3c+4)}$. By Lemma 2.14, $\langle Y, L_Y\rangle$ is recursive in $\mu^{((i-1)(3c+6)+3c+4)}$. By Corollary 4.27, $\mu^{((i-1)(3c+6)+3c+4+2)} = \mu^{(i(3c+6))}$ computes an ω-copy of J_δ. But this implies J_δ is represented in S_i. In particular, J_δ is represented in I_i. This contradicts the fact that δ is greater than any β such that J_β has a representation in I_{K-1}.

Case 2: $K = N+1$.

The analysis is similar to Case 1. γ_{N+1} is defined. Since R cannot be represented in any S_N, $\gamma_{N+1} < \beta_N$. Hence $J_{\gamma_{N+1}}$ is projectible, say to γ_i. There is an ω-copy of $J_{\gamma_{N+1}}$ recursive in $\mu^{(N(3c+6)+2c+4)}$. By the same argument as above, there is a copy of $\langle J_{\gamma_i}, A_{\gamma_{N+1}}\rangle$ recursive in $\mu^{((i-1)(3c+6)+3c+4)}$, which in turn yields a an ω-copy of $J_{\gamma_{N+1}}$ recursive in $\mu^{(i(3c+6))}$. This contradicts that $J_{\gamma_{N+1}}$ is not represented in any I_i, for $i \leq N$.

This is sufficient to complete the proof of Theorem 4.50. ☐

4.8. **Finishing the proof of Theorem 2.** We restate Theorem 2. Let $G(n)$ be the recursive function defined before the statement of Theorem 4.50 in Section 4.7.

Theorem 2. *For every $n \in \omega$,*

$$\mathsf{ZFC}_n^- \nvdash \text{“}\mathrm{NCR}_{G(n)} \textit{ is countable.”}$$

Proof. For any $n \geq 0$, the set $\mathcal{X}$ of canonical copies of standard J-structures $\langle J_{\rho^k_\alpha}, A^k_\alpha \rangle$ with $\rho^k_\alpha = 1$ is not countable in L_{β_n}. For suppose $f : \omega \twoheadrightarrow \mathcal{X}$ were a counting of $\mathcal{X}$ such that $f \in L_{\beta_n}$. We may assume f is given as a real. By the closure properties of L_{β_n}, $f' \in L_{\beta_n}$. Let $\gamma < \beta_n$ be the least ordinal such that $f \in J_{\gamma+1} \setminus J_\gamma$, and let m be such that f' is $\Sigma_m(J_\gamma)$. It follows that $\rho^m_\gamma = 1$. f' is computable in the canonical copy $\langle X^m_\alpha, M^m_\alpha \rangle$ of $\langle J_{\rho^m_\gamma}, A^m_\gamma \rangle$. It follows that $\langle X, M \rangle$ is not in the range of f. Since by Theorem 4.50, the set of canonical copies of standard J-structures is a subset of $\mathrm{NCR}_{G(n)}$. Therefore, $\mathrm{NCR}_{G(n)}$ is not countable in L_{β_n}. ☐

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