THE COMPLEXITY OF THE INDEX SETS OF $\aleph_0$-CATEGORICAL THEORIES AND OF EHRENFEUCHT THEORIES

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Abstract. We classify the computability-theoretic complexity of two index sets of classes of first-order theories: We show that the property of being an $\aleph_0$-categorical theory is Π^0_3-complete; and the property of being an Ehrenfeucht theory Π^1_1-complete. We also show that the property of having continuum many models is Σ^1_1-hard. Finally, as a corollary, we note that the properties of having only decidable models, and of having only computable models, are both Π^1_1-complete.

1. The Main Theorem

Measuring the complexity of mathematical notions is one of the main tasks of mathematical logic. Two of the main tools to classify complexity are provided by Kleene's arithmetical and analytical hierarchy. These two hierarchies provide convenient ways to determine the exact complexity of properties by various notions of *completeness*, and to give lower bounds on the complexity by various notions of *hardness*. (See, e.g., Kleene [1], Soare [10] or Odifreddi [4, 5] for the definitions.)

This paper will investigate the complexity of properties of a first-order theory, more precisely, the complexity of a countable first-order theory having a certain number of models. Recall that a theory is called $\aleph_0$-*categorical* if it has only one countable model up to isomorphism, and an *Ehrenfeucht theory* if it has more than one but only finitely many countable models up to isomorphism. In order to measure the complexity of these notions, we will use *decidable* first-order theories,

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i.e., sets T of first-order sentences closed under inference such that membership in T can be determined effectively.

The principal result of this paper is now the following

Main Theorem. *We classify the complexity of some properties of decidable first-order theories as follows:*

(1) *The property "T is an $\aleph_0$-categorical theory" is Π^0_3-complete.*
(2) *The property "T is an Ehrenfeucht theory" is Π^1_1-complete.*
(3) *The property "T has continuum many pairwise nonisomorphic models" is Σ^1_1-hard.*

In this paper, a *theory* will be a set of first-order sentences closed under inference (so that $T \vdash \sigma$ iff $\sigma \in T$ for any sentence σ).

The rest of this paper is devoted to the proof of our Main Theorem. In section 2, we will prove clause (1) of our Main Theorem. In section 3, we will show that the property of a theory being an Ehrenfeucht theory is Π^1_1, giving the upper bound for clause (2) of our Main Theorem. Finally, in section 4, we will prove a simultaneous reduction

$$(\Pi^1_1, \Sigma^1_1) \leq_m (\text{Ehrenfeucht}, \text{Continuum})$$

(where Ehrenfeucht and Continuum are (the index sets of) the properties of being an Ehrenfeucht theory or a theory having continuum many models, respectively). This reduction gives the lower bound for clause (2) and proves clause (3) of our Main Theorem.

2. The Proof for $\aleph_0$-Categoricity

We first show that "T's being $\aleph_0$-categorical" is Π^0_3 to obtain the upper bound: A theory T (with a characteristic function given by a partial computable function φ_e) is $\aleph_0$-categorical iff, by the Ryll-Nardzewski Theorem,

$$\varphi_e \text{ is a total function, } T \text{ is a complete consistent theory and}$$
$$\forall n\, (T \text{ has only finitely many } n\text{-types}),$$

i. e., iff

$$\begin{aligned} & e \in \text{Tot and } \forall \sigma\, (T \vdash \sigma \text{ implies } \sigma \in T) \text{ and} \\ & \forall \sigma\, (T \nvdash \sigma \wedge \neg\sigma) \text{ and } \forall \sigma\, (T \vdash \sigma \text{ or } T \vdash \neg\sigma) \text{ and} \\ & \forall n\, \exists m\, \exists \sigma_0, \dots, \sigma_m\, [\forall i \leq m\, (T \nvdash \sigma_i) \text{ and} \\ & \qquad \forall i \leq m\, \forall \tau\, (T \vdash \sigma_i \to \tau \text{ or } T \vdash \sigma_i \to \neg\tau)]. \end{aligned}$$

(where σ ranges over all first-order sentences, and σ_i and τ range over all first-order formulas in n variables $\overline{x}$ which we suppress above, respectively). Now, since "$T \vdash \sigma$" is Δ^0_1 for decidable theories, inspection shows that "T's being $\aleph_0$-categorical" is Π^0_3. (Recall here that we assume that T is closed under inference.)

In order to show that "T's being $\aleph_0$-categorical" is Π^0_3-complete, we present the following construction of a decidable theory T_e (uniformly in an index e) such that

$$\forall n\,(W_{f(n)} \text{ is finite}) \text{ iff } T_e \text{ is } \aleph_0\text{-categorical},$$

where $f = f_e$ is a computable function such that

$$\forall n\,(W_{f(n)} \text{ is finite})$$

is a Π^0_3-complete predicate (see, e.g., Soare [10, p. 68]). Without of loss of generality, we may assume that for every stage s, there is exactly one pair $\langle n, x\rangle$ such that x enters $W_{f(n)}$ at stage s. We denote this n by n_s.

The signature of our theory T_e now consists of relation symbols R^n_s (for all $n, s \in \omega$) where each R^n_s is an n-ary relation symbol. At stage s, we completely specify the relations R^n_s for all $n \in \omega$ as follows: For all $n \neq n_s$, we let the relation R^n_s be empty. For $n = n_s$, we let the relation R^n_s be "random" over all $R^{n_{s'}}_{s'}$ (for $s' < s$) in the sense that all finite extensions consistent with the theory enumerated before stage s (in the relation symbols $R^{n_{s'}}_{s'}$ for all n and all $s' < s$) are realized by the relation symbol $R^{n_s}_s$ added at stage s.

To verify that the construction yields the theory T_e with the desired properties, first assume that for some n, $W_{f(n)}$ is infinite. Then R^n_s is nonempty for infinitely many s, and in fact the reduct of T_e to these n-ary relations has continuum many consistent n-types, making T_e not $\aleph_0$-categorical.

On the other hand, assume that $W_{f(n)}$ is finite for all n. Then the n-type of any n-tuple $\overline{x}$ is determined by the finitely many nonempty relations of arity $\leq n$ satisfied by $\overline{x}$, so there are only finitely many n-types, and $\aleph_0$-categoricity follows.

In fact, it is not hard to see that the theory T_e admits elimination of quantifiers (effectively uniformly in e) and is decidable (again uniformly in e).

3. The Upper Bound for Ehrenfeucht theories

Proposition 1. *The set* $\{e : T_e$ *is an Ehrenfeucht theory*$\}$ *is* Π^1_1.

Proof. Proposition 1 follows from Sacks's [8] proof that every countable model of an Ehrenfeucht theory has a hyperarithmetic representation.

Suppose that T is a recursive, complete, first-order, Ehrenfeucht theory. From Sacks's theorem, each of the finitely many isomorphism types of models of T has a representative which is hyperarithmetically coded. In the same paper, Sacks shows that any model $\mathcal{A}$ of T completes its canonical Scott analysis in finitely many steps. Consequently, the canonical Scott sentence for $\mathcal{A}$ is uniformly hyperarithmetic in any representation A of $\mathcal{A}$. Further, if A and H are representations of countable first-order structures and have the same finite-rank canonical Scott sentence, then there is an isomorphism π between the models coded by A and H which is uniformly hyperarithmetically definable from A and H. See, for example, [3] in which Nadel analyzes of Scott sentences and isomorphisms between countable models within admissible sets. Thus, for T with the given properties, there are finitely many hyperarithmetic presentations $H_1, \dots, H_k$ of models of T such that for every representation A of a model of T, there is an isomorphism between the models coded by A one of the H_i's which is hyperarithmetic relative to A and $H_1, \dots, H_k$.

Conversely, for any recursive, complete, first-order theory T, if there is a hyperarithmetic finite sequence $H_1, \dots, H_k$ of representations of models of T such that for every A, if A codes a model of T then there is an isomorphism π between the model coded by A and that coded by one of the H_i's, then T is an Ehrenfeucht theory.

Now, the proposition follows. T_e's being a complete first-order theory is an arithmetic property and hence Π^1_1. By the Spector-Gandy Theorem, see Sacks [9], the Π^1_1 predicates are closed under existential quantification over the hyperarithmetic sets. Thus, the condition above, that there exist hyperarithmetic sets $H_1, \dots, H_k$, for all sets A, there exists π hyperarithmetic in $H_1, \dots, H_k$ and A, with arithmetically described properties, is a Π^1_1 condition. □

4. A Simultaneous Reduction for Ehrenfeucht Theories and Theories with Continuum Many Models

In this section, we will establish the simultaneous reduction

$$(\Pi^1_1, \Sigma^1_1) \leq_m (\mathrm{3Models}, \mathrm{Continuum}) \tag{1}$$

where 3Models and Continuum are (the index sets of) the properties of being an Ehrenfeucht theory with exactly three countable models, and a theory having continuum many countable models, respectively. Our proof will be based largely on Reed [7], which in turn used previous work of Peretyat'kin [6] and Millar [2]. (Since our proof uses the rather involved machinery of Reed [7] so heavily and mostly without changes, we will assume familiarity with this paper throughout the rest of the proof.)

We first observe that one can easily modify the proof of the Σ^1_1-completeness of the property of a computable tree $\mathrm{Tr} \subseteq \omega^{<\omega}$ having an infinite path to obtain a reduction

$$(\Pi^1_1, \Sigma^1_1) \leq_m (\mathrm{NoPath}, \mathrm{InfPath}) \tag{2}$$

where NoPath and InfPath are (the index sets of) the properties of being a computable tree $\mathrm{Tr} \subseteq \omega^{<\omega}$ having no infinite path, and having continuum many infinite paths, respectively.

We now compose the reduction from (2) with the reduction given in Reed [7]: Reed defines, for each computable tree $\mathrm{Tr} \subseteq \omega^{<\omega}$, and uniformly in an index e of Tr, a complete decidable theory T_e "coding" the tree Tr into a "dense tree". (Actually, Reed only defines T_e for trees Tr having exactly one infinite path, but his definition in Part II of [7] can be applied to any computable $\mathrm{Tr} \subseteq \omega^{<\omega}$ and always yields a complete decidable theory T_e.)

Checking over Reed's analysis of 1-types over T_e in Part III of [7], one can now easily verify the following: If Tr has no infinite path then T_e has exactly three countable models, namely, the countable computable models omitting the type $\Gamma^*(x)$ (since Tr contains no infinite paths). On the other hand, if Tr has infinitely many infinite paths then T_e admits a partial type $\Gamma^*_f = \{c_\xi < x \mid \xi \subset f\}$ for each infinite path $f \in [\mathrm{Tr}]$, and by an argument analogous to that for Corollary 15.1 in Reed [7] (which shows that the type $\Delta(x)$ can be realized without the type $\Gamma^*(x)$ being realized), for any two distinct paths $f, g \in [\mathrm{Tr}]$, the partial types Γ^*_f and Γ^*_g can be realized independently of each other. Thus T_e has continuum many types and so also continuum many countable models.

To finish off, we note an easy corollary of our proof as well as two remarks:

Corollary 2. *The property of a decidable theory having only decidable models is Π^1_1-hard, as is the property of a decidable theory having only computable models.*

Remark 3.
(1) Our proof actually shows that the property of being an Ehrenfeucht theory with exactly three models is already Π^1_1-complete.
(2) The first-order language used in our proof is not fixed; i. e., the language depends on the tree Tr and thus on the index e used for the construction. However, we can simply add constant symbols c_η (for $\eta \in \omega^{<\omega} - T$) denoting a fixed element of our model, and empty relations E^η_ξ, etc. (for η or $\xi \in \omega^{<\omega} - T$), to achieve a fixed language.

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