

HOW TO APPLY SERRE'S CRITERION TO PROVE PRIMENESS

Shuchao Bi
January, 2009

ABSTRACT. In this write-up, I'll present some examples to illustrate how to apply Serre's criterion to prove primeness. The most important examples are Examples 5 and 6 which I believe will lead to a new proof of that generic determinantal ideals are prime.

Theorem 1. (Serre's criterion) A Noetherian ring R is a direct product of normal domains iff the following two conditions are satisfied:

S2: Every associated prime of a principal ideal generated by a nonzerodivisor in R is of codimension 1; every associated prime of 0 is of codimension 0.

R1: Every localization of R at a codimension-1 prime is a discrete valuation ring; every localization of R at a codimension-0 prime is a field.

The following theorem is a combination of Serre's criterion with Jacobian criterion.

Theorem 2. Let $R = k[x_1, \dots, x_r][U^{-1}]/I$ be a localization of affine ring over a perfect field k . Suppose that $I = (f_1, \dots, f_n)$ has codimension c . Let J be the ideal generated by the $c \times c$ minors of the Jacobian matrix $(\partial f_i / \partial x_j)$, taken modulo I . If R is Cohen-Macaulay, then

- R is reduced iff J has codimension ≥ 1 in R .
- R is a direct product of domains iff condition a holds and R_P is a domain for every prime P of codimension ≤ 1 .
- R is a direct product of normal domains iff J has codimension ≥ 2 in R .

Using these criteria, one can only show that R is a product of domains. To show that R is actually a domain, we need some further information. For example, we might know that R can not have idempotents because $\text{Spec}R$ is connected or r is local (so that any nonzero idempotent is a unit) or graded (so that any idempotent has degree 0) with degree-0 part a domain.

In example 3 and 4 below, assume k is a perfect field.

Example 1. $R = k[x, y, z, w]/(xy - zw)$ is a normal domain.

First R is Cohen-Macaulay since it is a hypersurface. The Jacobian matrix of $(xy - zw)$ is $(y, x, -w, -z)$. Then $J = (x, y, z, w)/(xy - zw)$, it has codimension ≥ 3 in R because $x, y, z + w$ is a R -regular sequence in J . By theorem 2, R is a product of normal domains, but R is graded with degree zero part a field, so R itself is a normal domain. Note, $\text{codim}(J)$ actually equals to 3, because if we have $\text{codim}(J) > 3$, then (x, y, z, w) will have codimension greater than 4 ((which is nonsense)) as $xy - zw$ is not contain in any minimal prime.

Of course, we can prove that $(xy - zw)$ is prime by elementary method. For example, first show that $xy - zw$ is irreducible, then since $k[x, y, z, w]$ is factorial, $(xy - zw)$ is prime as we hoped.

Note $xy - zw$ is the determinant of $\begin{pmatrix} x & z \\ w & y \end{pmatrix}$.

We will generalize this idea in Examples 5 and 6.

Example 2. $R = k[x, y, z]/(xy - z^2)$ is a normal domain. The proof is the same as in Example 1.

In example 3 and 4 below, assume k is algebraically closed with characteristic not 2.

Example 3. $R = k[x_1, \dots, x_n]/(x_1^2 + \dots + x_r^2)$

R is Cohen-Macaulay and the Jacobian matrix is $(2x_1, \dots, 2x_r)$, so $J = (x_1, \dots, x_r)/(x_1^2 + \dots + x_r^2)$. We claim J has codimension $r - 1$ in R . To prove this, first notice that x_1, \dots, x_r is a regular R -sequence, so $\text{codim}(J) \geq r - 1$ in R . On the other hand, (x_1, \dots, x_r) has codimension r in $k[x_1, \dots, x_n]$, and $x_1^2 + \dots + x_r^2$ is not a nilpotent element, so J has codimension $\leq r - 1$ in R .

Apply theorem 2, we get that R is reduced iff $r \geq 2$ and R is a normal domain iff $r \geq 3$.

Example 4. Let $I = (Q_1, Q_2) \subset k[x_0, x_1, x_2, x_3]$ where

$$Q_1 = x_0^2 + x_1^2 + x_2^2 + x_3^2$$

$$Q_2 = a_0x_0^2 + a_1x_1^2 + a_2x_2^2 + a_3x_3^2 \text{ where } a_i \in k.$$

Since Q_1 is irreducible and $k[x_0, x_1, x_2, x_3]$ is factorial, (Q_1) is prime. (Or we can see this directly in Example 3.) So Q_1, Q_2 is a regular sequence as long as Q_2 is not a multiple of Q_1 . Assume that not all a_i s are equal, then $R = k[x_0, x_1, x_2, x_3]/(Q_1, Q_2)$ is Cohen-Macaulay, so we can apply theorem 2.

The Jacobian matrix is

$$\begin{pmatrix} 2x_0 & 2x_1 & 2x_2 & 2x_3 \\ 2a_0x_0 & 2a_1x_1 & 2a_2x_2 & 2a_3x_3 \end{pmatrix}$$

Write J' for the ideal generated by 2×2 minors of this Jacobian matrix. Then J' is generated by $(a_i - a_j)x_i x_j$ for all $i \neq j$.

If two of a_i s are equal, say $a_0 = a_1$, then

$$Q_2 - a_0 Q_1 = (a_2 - a_0)x_2^2 + (a_3 - a_0)x_3^2$$

which factors as

$$(b_2x_2 + b_3x_3)(b_2x_2 - b_3x_3)$$

where b_i is a square root of $a_i - a_0$. These factors are linear and thus neither of them is contained in I , so in this case, I is not a prime.

Assume all a_i are distinct, in this case $J' = (x_0x_1, x_0x_2, x_0x_3, x_1x_2, x_1x_3, x_2x_3)$. The primary decomposition of J' is

$$J' = (x_0, x_1, x_2) \cap (x_0, x_1, x_3) \cap (x_0, x_2, x_3) \cap (x_1, x_2, x_3).$$

So the minimal primes of J' are primes generated by any 3 of the variables. Q_1 is not contained in any of these primes, so the minimal primes of $J' + Q_1$ strictly contains one of the minimal primes of J' , and thus $J' + Q_1$, hence $J' + I$ has codimension 4. It follows that $J = (J' + I)/I$ has codimension 2 because $k[x_0, x_1, x_2, x_3]$ is catenary. Apply theorem 2, $R = k[x_0, x_1, x_2, x_3]/(Q_1, Q_2)$ is a normal domain.

In summary, the ideal (Q_1, Q_2) has codimension 2 iff not all a_i are equal; it is prime iff all a_i are distinct.

Example 5. Let $S = k[x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, \dots, x_{n1}, \dots, x_{nn}]$ and

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \cdots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}.$$

Let $I = (\det(X))$ and $R = S/I$, we want to show $\det(X)$ is an irreducible polynomial.

First R is Cohen-Macaulay since it is a quotient of a polynomial ring by a regular sequence (length 1). The Jacobian matrix is the vector of all $n - 1 \times n - 1$ minors of X . So the Jacobian ideal is generated by all $n - 1 \times n - 1$ minors of X which is denoted by I_{n-1} . Northcott and Eagon proved that $\text{codim}(I_{n-1})$ is $2 \times 2 = 4$, so the quotient ideal I_{n-1}/I has codimension 3 in R . By theorem 2, R is a product of normal domains, but R is graded with degree zero part a field, so R itself is a normal domain. We are done.

Example 6. Generalize Example 5, let $S = k[x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, \dots, x_{m1}, \dots, x_{mn}]$ and

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \cdots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix}$$

Let I_k denote the ideal generated by all $k \times k$ minors of X and $R = S/I_k$. We want to show I_k is a prime ideal.

The fact that R is Cohen-Macaulay is also proved by Northcott and Eagon using Northcott-Eagon complex. to be continued...

It seems hard to find out the codimension of the jacobian ideal. I did the cases $m = 2, n = 3, k = 2$ and $m = 3, n = 3, k = 2$, Serre's criterion works for both of them.

Non-example 1. $R = k[x, y]/(y^2 - x^3)$ is not a normal domain.

First R is Cohen-Macaulay since it is a hypersurface. The Jacobian matrix of $(y^2 - x^3)$ is $(-3x^2, 2y)$. Then $J = (x^2, y)/(y^2 - x^3)$, it has codimension ≤ 1 in R because if $\text{codim}(J) > 1$, then (x, y) will have codimension greater than 2 (which is nonsense) as $y^2 - x^3$ is not contained in any minimal prime. If R is a normal domain, then by theorem 2, $\text{codim}(J) \geq 2$ contradiction.

From another point of view, If $k[x, y]/(y^2 - x^3)$ is normal, then because it has dimension 1, every localization at maximal ideal is a regular local ring, i.e., the curve is smooth. But we know from the Jacobian criterion that $(0, 0)$ is singular point.

Non-example 2. $R = k[x, y]/(y^2 - x^2 - x^3)$ is not a normal domain. The reason is the same as in Non-example 1.

REFERENCES

David Eisenbud, Commutative Algebra with a View Toward Algebraic Geometry.

Qing Liu, Algebraic Geometry and Arithmetic Curves.