Elementary Operad Theory

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Abstract

Operad theory, developed by J. P. May in algebraic topology in 1970s, found its usage in many other branches of mathematics, such as algebra and algebraic geometry. It generalizes those original algebraic types which admit only some binary operations. Moreover, it gives a natural filtration to each free algebra over an operad, which makes it easier to calculate some invariants of algebras over it. In this paper we try to summarize the basic theory about operads and give some of its first properties.

It seems that the most general definition of operad is operads in an arbitrary symmetric monoidal category. We first define the action of a group G on an object in an arbitrary category (not necessarily monoidal) $\mathfrak{S}$.

Definition 1. *Let G be a group and $x \in \mathfrak{S}$ an object. A left action of G on x is a group homomorphism $G \to Aut_{\mathfrak{S}}(x,x)$, where $Aut_{\mathfrak{S}}(x,x)$ is the group of units in the monoid* $\hom_{\mathfrak{S}}(x,x)$ *under composition. A right action of G on x is a function $G \to Aut_{\mathfrak{S}}(x,x)$ which becomes a group homomorphism when composed with the inverse map $G \to G$.*

It is not hard to see that we can generalize to define a left action of a monoid G in $\mathfrak{Set}$ on an object x to be a monoid morphism $G \to hom(x,x)$, where the hom-set is a monoid under composition of morphisms—-the image of G will certainly fall into its subgroup of units when G itself is a group. Now we turn to the definition of operad.

Definition 2. *An operad $\mathcal{C}$ in a symmetric monoidal category $(\mathfrak{S}, \otimes, \kappa)$ is a collection $\mathcal{C}(j)$ of objects in $\mathfrak{S}$ indexed by $j \in \mathbb{N}$ together with a right action by the permutation group Σ_j on $\mathcal{C}(j)$ for any j, a map $\eta : \kappa \to \mathcal{C}(1)$ and maps*

$$\gamma : \mathcal{C}(k) \otimes \bigotimes_{r=1}^{k} \mathcal{C}(j_r) \to \mathcal{C}(j)$$

for $k \geq 1$ and $j_r \geq 0$, where $\sum j_r = j$, which subjects to the following associative, unital and equivariant conditions.

1. *Associativity. The following diagram commutes:*

$$\begin{array}{ccc}
\mathcal{C}(k) \otimes (\bigotimes_{r=1}^{k} \mathcal{C}(j_r)) \otimes (\bigotimes_{t=1}^{j} \mathcal{C}(i_t)) & \xrightarrow{\gamma\otimes id} & \mathcal{C}(j) \otimes (\bigotimes_{t=1}^{j} \mathcal{C}(i_t)) \\
\downarrow{\scriptstyle shuffle} & & \downarrow{\scriptstyle \gamma} \\
\mathcal{C}(k) \otimes (\bigotimes_{r=1}^{k} (\mathcal{C}(j_r) \otimes (\bigotimes_{q=1}^{j_r} \mathcal{C}(i_{g_{r-1}+q})))) & & \mathcal{C}(i) \\
\downarrow{\scriptstyle id\otimes(\otimes_r \gamma)} & & \downarrow{\scriptstyle id} \\
\mathcal{C}(k) \otimes (\bigotimes_{r=1}^{k} \mathcal{C}(h_r)) & \xrightarrow{\gamma} & \mathcal{C}(i)
\end{array},$$

where $\sum_r j_r = j$, $\sum_t i_t = i$, and we set $g_r = j_1 + \cdots + j_r$ and $h_r = i_{g_{r-1}+1} + \cdots + i_{g_r}$ for $1 \leq s \leq k$;

2. *Unitary. The following diagrams commute:*

$$\begin{array}{ccc} \mathcal{C}(k)\otimes(\kappa)^k & & \\ {\scriptstyle id\otimes\eta^k}\downarrow & \searrow^{\cong} & \\ \mathcal{C}(k)\otimes\mathcal{C}(1)^k & \xrightarrow[\gamma]{} & \mathcal{C}(k) \end{array} \qquad \begin{array}{ccc} \kappa\otimes\mathcal{C}(j) & & \\ {\scriptstyle \eta\otimes id}\downarrow & \searrow^{\cong} & \\ \mathcal{C}(1)\otimes\mathcal{C}(j) & \xrightarrow[\gamma]{} & \mathcal{C}(j) \end{array} \quad ;$$

3. *Equivariance. The following diagrams commute:*

$$\begin{array}{ccc} \mathcal{C}(k)\otimes\bigotimes_{r=1}^{k}\mathcal{C}(j_r) & \xrightarrow{\sigma\otimes\sigma^{-1}} & \mathcal{C}(k)\otimes\bigotimes_{r=1}^{k}\mathcal{C}(j_{\sigma(r)}) \\ \gamma\downarrow & & \gamma\downarrow \\ \mathcal{C}(j) & \xrightarrow{\sigma(j_\sigma(1),\cdots,j_\sigma(k))} & \mathcal{C}(j) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{C}(k)\otimes\bigotimes_{r=1}^{k}\mathcal{C}(j_r) & \xrightarrow{id\otimes\tau_1\otimes\cdots\otimes\tau_k} & \mathcal{C}(k)\otimes\bigotimes_{r=1}^{k}\mathcal{C}(j_r) \\ \gamma\downarrow & & \gamma\downarrow \\ \mathcal{C}(j) & \xrightarrow{\tau_1\oplus\cdots\oplus\tau_k} & \mathcal{C}(j) \end{array} \quad ,$$

where $\sigma \in \Sigma_k, \tau_s \in \Sigma_{j_s}, j = \sum_{r=1}^{k} j_r$, the permutation $\sigma(j_1, \cdots, j_k) \in \Sigma_j$ permutes k blocks of letter as σ permutes k letters, and $\tau_1 \oplus \cdots \oplus \tau_k \in \Sigma_j$ is the block sum.

A morphism f between two operads $\mathcal{C}$ and $\mathcal{D}$ in $\mathfrak{S}$ is a sequence $f(j) : (C)(j) \to \mathcal{D}(j)$ of morphisms for each $j \in \mathbb{N}$, such that all the evident diagrams commute.

When concerned with unital algebraic types, the "unit element" $1 \in A$ is given by a map $\kappa \to A$, so we usually assume in addition that $\mathcal{C}(0) = \kappa$. When concerned with algebraic types without unit element we assume instead that $\mathcal{C}(0) = \emptyset$, the initial object in $\mathfrak{S}$.

This abstract definition appears difficult to understand, but not so if one thinks of those $\mathcal{C}(j)$ as the objects of parameters for j-ary operations that accept j inputs and produce one output. Then the action by the symmetric group means to permute the order of the inputs and the maps γ means to compose different operations. The following example makes this precise.

Example 1. Assume $\mathfrak{S}$ has an internal Hom functor, i.e., $\mathfrak{S}$ is enriched into itself via some $H : \mathfrak{S} \times \mathfrak{S} \to \mathfrak{S}$ and $H(s, -)$ provides a right adjoint for $- \otimes s$. Given any object $x \in \mathfrak{S}$, we define the endomorphism operad $End(x)$ of x by setting

$$End(x)(j) = H(x^j, x),$$

where x^j denotes the j-fold tensor product of x. The map $\eta : \kappa \to H(x, x)$ is given in the definition of an enrichment, which plays the role of the identity morphism of x; the right actions by symmetric groups are given by their left actions on tensor products, and the maps γ are given by

$$H(x^k, x) \otimes \bigotimes_{r=1}^{k} H(x^{j_r}, x) \to H(x^k, x) \otimes H(x^j, x^k) \to H(x^j, x).$$

Since $End(x)$ is an operad associated with the object x, a morphism of operads $\mathcal{C} \to End(x)$ can "inject" the information of operations in $\mathcal{C}$ into the object x via the given morphism. We call such an object x together with a specified morphism an algebra over the operad $\mathcal{C}$. But this is too restrictive since it requires the category $\mathfrak{S}$ to be endowed with an internal Hom functor. So we generalize to give the following definition.

Definition 3. *Let $\mathcal{C}$ be an operad in $\mathfrak{S}$. A $\mathcal{C}$-algebra is an object $x \in \mathfrak{S}$ together with maps*

$$\theta : \mathcal{C}(j) \otimes x^j \to x$$

for $j \geq 0$ that are associative, unital and equivariant:

1. *The following associativity diagrams commute, where $j = \sum_{r=1}^{k} j_r$:*

$$\begin{array}{ccc} \mathcal{C}(k) \otimes (\bigotimes_{r=1}^{k} \mathcal{C}(j_r)) \otimes x^j & \xrightarrow{\gamma \otimes id} & \mathcal{C}(j) \otimes x^j \\ {\scriptstyle shuffle}\downarrow & & \downarrow{\scriptstyle \theta} \\ \mathcal{C}(k) \otimes (\bigotimes_{r=1}^{k} (\mathcal{C}(j_r) \otimes x^{j_r})) & & x \\ {\scriptstyle id \otimes \theta^k}\downarrow & & \downarrow{\scriptstyle id} \\ \mathcal{C}(k) \otimes x^k & \xrightarrow{\theta} & x \end{array} ;$$

2. *The following unit diagram commutes:*

$$\begin{array}{ccc} \kappa \otimes x & & \\ {\scriptstyle \eta \otimes id}\downarrow & \searrow{\scriptstyle \cong} & \\ \mathcal{C}(1) \otimes x & \xrightarrow[\theta]{} & x \end{array} ;$$

3. *The following equivariance diagrams commute, where $\sigma \in \Sigma_j$:*

$$\begin{array}{ccc} \mathcal{C}(j) \otimes x^j & \xrightarrow{\sigma \otimes \sigma^{-1}} & \mathcal{C}(j) \otimes x^j \\ & {\scriptstyle \theta}\searrow \quad \swarrow{\scriptstyle \theta} & \\ & x & \end{array} .$$

A morphism between two algebras $x, y \in \mathfrak{S}$ over $\mathcal{C}$ is a morphism from x to y which makes all evident diagrams commute.

It can be seen easily that if the category $\mathfrak{S}$ admits an internal Hom functor H, then the definition of algebras over the operad $\mathcal{C}$ can be written in adjoint form as a morphism of operads $\mathcal{C} \to End(x)$, and a morphism between two algebras x and y over $\mathcal{C}$ is just a morphism from x to y such that the induced morphism $End(x) \to End(y)$ of operads is a morphism under $\mathcal{C}$, i.e., it makes the triangle commute:

$$\begin{array}{ccc} & & End(x) \\ & \nearrow & \downarrow \\ \mathcal{C} & & \\ & \searrow & \\ & & End(y) \end{array} .$$

Lemma 1. *An operad morphism $\omega : \mathcal{C} \to \mathcal{D}$ yields a functor $\omega^* : \mathcal{D}\text{-}\mathfrak{Alg} \to \mathcal{C}\text{-}\mathfrak{Alg}$ by pullback along ω.*

Like a monoid can use its multiplication structure to act on other objects in order to make them modules, an algebra over an operad can also have modules on which it acts. But now the algebra given has various n-ary operations, so it acts on a module using all these structures in a compatible way.

Definition 4. *Let $\mathcal{C}$ be an operad in $\mathfrak{S}$ and x be a $\mathcal{C}$-algebra. An x-module is an object m together with maps*

$$\lambda : \mathcal{C}(j) \otimes x^{j-1} \otimes m \to m$$

for any $j \geq 1$, which are associative, unital and equivariant:

1. *The following associativity diagram commutes, where $j = \sum j_r$:*

$$\begin{array}{ccc}
(\mathcal{C}(k) \otimes (\bigotimes_{r=1}^{k} \mathcal{C}(j_r))) \otimes x^{j-1} \otimes m & \xrightarrow{\gamma\otimes id} & \mathcal{C}(j) \otimes x^{j-1} \otimes m \\
\downarrow{\scriptstyle shuffle} & & \downarrow{\scriptstyle \lambda} \\
\mathcal{C}(k) \otimes (\bigotimes_{r=1}^{k-1} (\mathcal{C}(j_r) \otimes x^{j_r})) \otimes (\mathcal{C}(j_k) \otimes x^{j_k-1} \otimes m) & & m \\
\downarrow{\scriptstyle id\otimes\theta^{k-1}\otimes\lambda} & & \downarrow{\scriptstyle id} \\
\mathcal{C}(k) \otimes x^{k-1} \otimes m & \xrightarrow{\lambda} & m
\end{array};$$

2. *The following unit diagram commutes:*

$$\begin{array}{ccc}
\kappa \otimes m & & \\
\downarrow{\scriptstyle \eta\otimes id} & \searrow{\scriptstyle \cong} & \\
\mathcal{C}(1) \otimes m & \xrightarrow[\lambda]{} & m
\end{array};$$

3. *The following equivariance diagram commutes, where $\sigma \in \Sigma_{j-1} \subset \Sigma_j$, the inclusion realizing Σ_{j-1} as a subgroup of Σ_j fixing the last item:*

$$\begin{array}{ccc}
\mathcal{C}(j) \otimes x^{j-1} \otimes m & \xrightarrow{\sigma\otimes\sigma^{-1}\otimes id} & \mathcal{C}(j) \otimes x^{j-1} \otimes m \\
& {\scriptstyle \lambda}\searrow \quad \swarrow{\scriptstyle \lambda} & \\
& m &
\end{array}.$$

A morphism of two modules m and m' over a $\mathcal{C}$-algebra x is a morphism from m to m' in $\mathfrak{S}$ making all the diagrams involving λ's commute.

Next we explore the relation between operads and monads in $\mathfrak{S}$.

Definition 5. *Let $\mathfrak{S}$ be a category(not necessarily monoidal). Then the functor category* $\hom_{\mathfrak{Cat}}(\mathfrak{S}, \mathfrak{S}) = \mathfrak{S}^{\mathfrak{S}}$ *together with composition of endofunctors form a monoidal category. We define a monad in $\mathfrak{S}$ to be a monoid (T, μ, η) in this monoidal category. A pair $(x \in \mathfrak{S}, f : \mathrm{T}x \to x)$ is called an algebra over the monad (T, μ, η) if the following two diagrams commute:*

$$\begin{array}{ccc}
\mathrm{T}^2 x & \xrightarrow{\mu_x} & \mathrm{T}x \\
\downarrow{\scriptstyle \mathrm{T}f} & & \downarrow{\scriptstyle f} \\
\mathrm{T}x & \xrightarrow{f} & x
\end{array}
\qquad
\begin{array}{ccc}
& \mathrm{T}x & \\
{\scriptstyle \eta_x}\nearrow & & \searrow{\scriptstyle f} \\
x & \xrightarrow{id} & x
\end{array}.$$

A morphism of T*-algebras from* (x, f) *to* (y, g) *is a morphism* $s : x \to y$ *such that*

$$\begin{array}{ccc} \mathrm{T}x & \xrightarrow{\mathrm{T}s} & \mathrm{T}y \\ f\downarrow & & g\downarrow \\ x & \xrightarrow{s} & y \end{array}$$

commutes.

As expected, the free algebra over a monad (T, μ, η) generated by x exist for any object $x \in \mathfrak{S}$.

Proposition 2. *The forgetful functor* $U : \mathrm{T}\text{-}\mathfrak{Alg} \to \mathfrak{S}$ *sending* (x, f) *to* x *and* $s : (x, f) \to (y, g)$ *to* $s : x \to y$ *has* $F : \mathfrak{S} \to \mathrm{T}\text{-}\mathfrak{Alg}$ *as its left adjoint functor, where* F *is given by*

$$Fx = (\mathrm{T}x, \mu_x : \mathrm{T}^2x \to \mathrm{T}x)$$

and

$$F(s : x \to y) = (\mathrm{T}s : \mathrm{T}x \to \mathrm{T}y).$$

Proof. See [Mac] p140. $\square$

Before talking about the relation between operads and monads, we first give an alternative description of operads, which is more economical. We have to assume from now on that the base category $\mathfrak{S}$ is finitely cocomplete and the tensor product is distributive over the coproducts, i.e., the natural map

$$\bigoplus_i (a \otimes b_i) \to a \otimes (\bigoplus_i b_i)$$

induced by those

$$a \otimes b_i \to a \otimes (\bigoplus_i b_i)$$

is always an isomorphism.

Notation 1. *Let* (c, μ, η) *be a monoid in a monoidal category* $\mathfrak{S}$*, and let* $a, b \in \mathfrak{S}$ *be a right module and a left module over* c*, respectively. Denote by* $a \otimes_c b$ *the coequalizer of the two maps*

$$a \otimes c \otimes b \rightrightarrows a \otimes b$$

given by the two actions $a \otimes c \to a$ *and* $c \otimes b \to b$*.*

Lemma 3. *Assume* $\mathfrak{S}$ *is locally small. Let* $(\mathfrak{Set}, \times, 1)$ *be the monoidal category of small sets with tensor products the cartesian products and* $1 = \{\emptyset\}$ *a one-point set as identity object. Then the two functors*

$$\hom_{\mathfrak{S}}(\kappa, \square) : \mathfrak{S} \to \mathfrak{Set}$$

and

$$\coprod_{\square} \kappa : \mathfrak{Set} \to \mathfrak{S}$$

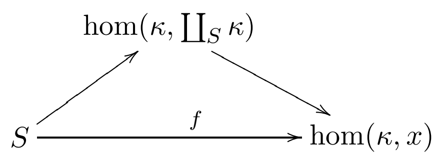
are both monoidal functors, so in particular they induce functors on the category of monoids $\mathbf{Mon}(\mathfrak{S})$ *and* $\mathbf{Mon}(\mathfrak{Set})$*. Moreover, they are adjoint:*

$$\coprod_{\square} \kappa \dashv \hom(\kappa, \square).$$

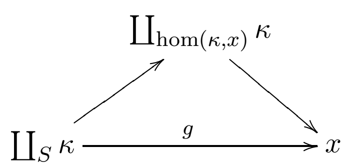
Proof. $\hom(\kappa, x) \times \hom(\kappa, y) \to \hom(\kappa \otimes \kappa, x \otimes y) \cong \hom(\kappa, x \otimes y)$ given by the isomorphism $\kappa \otimes \kappa \cong \kappa$ shows how the tensor products behave under the functor $\hom(\kappa, \Box)$. Also there's a natural map $1 \to \hom(\kappa, \kappa)$ mapping $\emptyset$ to the identity morphism on κ. Let S, T be two sets. Then we have natural maps

$$\coprod_S \kappa \otimes \coprod_T \kappa \to \coprod_S (\kappa \otimes \coprod_T \kappa) \to \coprod_S (\coprod_T \kappa \otimes \kappa) \cong \coprod_{S \times T} \kappa.$$

Also we have $\kappa \cong \coprod_1 \kappa$. Then one verifies easily these natural maps satisfies the conditions in the definition of a relaxed monoidal functor. For the adjointness, first we establish the two natural transformations $id \to \hom(\kappa, \coprod_\Box \kappa)$ and $\coprod_{\hom(\kappa, \Box)} \kappa \to id$, and then show their universal properties, although showing one of them is enough. For any set $S \in \mathfrak{Set}$, the map $S \to \hom(\kappa, \coprod_S \kappa)$ is given by sending an element $s \in S$ to the inclusion $\iota_s : \kappa \rightarrowtail \coprod_S \kappa$ of κ into the s-th copy of κ. For any set function $f : S \to \hom(\kappa, x)$ for some $x \in \mathfrak{S}$, there exists a unique morphism $\coprod_S \kappa \to x$, the one induced by $f(s) : \kappa_s \to x$, where by κ_s we mean the s-th copy of κ, such that the triangle



commutes. For any object $x \in \mathfrak{S}$, the natural map $\coprod_{\hom(\kappa, x)} \kappa \to x$ is induced by $t : \kappa_t \to x$ for each $t \in \hom(\kappa, x)$. For any morphism $g : \coprod_S \kappa \to x$, there exists a uniquely function $S \to \hom(\kappa, x)$, given by $s \in S \mapsto g \circ \iota_s$, where ι_s is the s-th inclusion $\kappa \rightarrowtail \coprod_S \kappa$, such that the triangle



commutes. $\Box$

Sometimes we use $\kappa[S]$ to denote $\coprod_S \kappa$. So if G is a monoid in $\mathfrak{Set}$, $\kappa[G]$ is a monoid in $\mathfrak{S}$. In particular, $\kappa[\Sigma_j]$ is a monoid, for any $j \geq 0$.

Lemma 4. *Let G be a monoid in $\mathfrak{Set}$, $x \in \mathfrak{S}$. Then to give a left action of G on x in the sense of Definition 1 is equivalent to giving a left action of the monoid $\kappa[G]$ in $\mathfrak{S}$ on x. When G is a group, the above also holds for right actions.*

Proof. Given a left action $\kappa[G] \otimes x \to x$ of $\kappa[G]$ on x, we have maps

$$\coprod_G x \cong \coprod_G (\kappa \otimes x) = \kappa[G] \otimes x \to x,$$

the composition being denoted by φ. Then we get a left action $G \to \hom(x, x)$ in the sense of Definition 1 by sending $g \in G$ to $\varphi \circ \iota_g : x \to x$, where $\iota_g : x \rightarrowtail \coprod_G x$ is the g-th embedding. One checks this is indeed a morphism of monoids. Conversely, given a left action $\psi : G \to \hom(x, x)$, we have

$$\kappa[G] \otimes x \cong \coprod_G x \to x$$

which is induced by $\psi(g) : x_g \to x$. Similarly for right actions when G is a group. $\Box$

Definition 6. *Let Σ be the category whose objects are the finite ordinal numbers $n = \{n-1\} \cup (n-1) = \{0, 1, \cdots, n-1\}$, with bijections as their morphisms. A Σ-object in $\mathfrak{S}$ is defined to be a contravariant functor $\mathcal{C} : \Sigma^{op} \to \mathfrak{S}$. So the natural left action of Σ_j on the set j induces a right action on $\mathcal{C}(j)$. By convention we set $\mathcal{C}(0) = \kappa$. We want to make the category $\mathfrak{S}^{\Sigma^{op}}$ of Σ-objects in $\mathfrak{S}$ into a monoidal category such that the category of operads in $\mathfrak{S}$ is just the category of monoids in this monoidal category. The tensor product on Σ-objects $\boxtimes$ is defined to be*

$$(\mathcal{B} \boxtimes \mathcal{C})(j) = \coprod_{k, j_1, \cdots, j_k} \mathcal{B}(k) \otimes_{\kappa[\Sigma_k]} ((\bigotimes_{r=1}^{k} \mathcal{C}(j_r)) \otimes_{\kappa[\prod_{r=1}^{k} \Sigma_{j_r}]} \kappa[\Sigma_j]),$$

where $k \geq 0, j_r \geq 0$, and $\sum_{r=1}^{k} j_r = j$. The natural map $\coprod_{\prod_r \Sigma_{j_r}} \kappa \to \bigotimes_r \coprod_{\Sigma_{j_r}} \kappa$ is an isomorphism by assumption, and it's the inverse of the natural map $\bigotimes_r \coprod_{\Sigma_{j_r}} \kappa \to \kappa[\prod_r \Sigma_{j_r}]$ given in the definition of a relaxed monoidal functor. The right action of Σ_{j_r} on $\mathcal{C}(j_r)$ yields by lemma 3 a right action of $\kappa[\Sigma_{j_r}]$ on $\mathcal{C}(j_r)$, for each r, hence they induce a right action of $\bigotimes_{r=1}^{k} \kappa[\Sigma_{j_r}]$ on $\bigotimes_{r=1}^{k} \mathcal{C}(j_r)$, since $\mathfrak{S}$ is symmetric. But by assumption the natural map

$$\kappa[\prod_r \Sigma_{j_r}] = \coprod_{\prod_r \Sigma_{j_r}} \kappa \to \bigotimes_r \coprod_{\Sigma_{j_r}} \kappa = \bigotimes_r \kappa[\Sigma_{j_r}]$$

is an isomorphism, so we have a right action of $\kappa[\prod_r \Sigma_{j_r}]$ on $\bigotimes_{r=1}^{k} \mathcal{C}(j_r)$ via this isomorphism. The left action of $\kappa[\prod_r \Sigma_{j_r}]$ on $\kappa[\Sigma_j]$ is induced by the map $\prod_r \Sigma_{j_r} \to \Sigma_j$ sending $(\tau_{j_1}, \cdots, \tau_{j_k})$ to $\bigoplus_r \tau_{j_r}$, say, multiplying $\bigoplus_r \tau_{j_r}$ to Σ_j on the left. The left action of Σ_k on $(\bigotimes_{r=1}^{k} \mathcal{C}(j_r)) \otimes_{\kappa[\prod_{r=1}^{k} \Sigma_{j_r}]} \kappa[\Sigma_j]$ is given by $\cdots$. This explains the notation in the formula defining $\boxtimes$. This $\boxtimes$ is associative up to coherent natural isomorphism, and has the Σ-object $\mathcal{I}$ specified by $\mathcal{I}(1) = \kappa$, $\mathcal{I}(j) = \emptyset$(an initial object in $\mathfrak{S}$, exists since $\mathfrak{S}$ is finitely cocomplete) for $j \neq 1$ as its two-sided unit. So the category $\mathfrak{S}^{\Sigma^{op}}$ of Σ-objects together with $\boxtimes$ and $\mathcal{I}$ is a monoidal category.

Lemma 5. *The category of operads in $\mathfrak{S}$ is isomorphic to the category $\mathbf{Mon}(\mathfrak{S}^{\Sigma^{op}})$ of monoids in the category $(\mathfrak{S}^{\Sigma^{op}}, \boxtimes, \mathcal{I})$ of Σ-objects in $\mathfrak{S}$.*

Proof. The associativity diagram in the definition of operad is an arbitrary component of the associativity diagram in the definition of monoids:

$$\begin{array}{ccc} \mathcal{C}^{\boxtimes 3}(i) & \xrightarrow{\mu \boxtimes id} & \mathcal{C}^{\boxtimes 2}(i) \\ {\scriptstyle id \boxtimes \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathcal{C}^{\boxtimes 2}(i) & \xrightarrow{\mu} & \mathcal{C}(i) \end{array},$$

whose commutativity will together imply the associativity of the multiplication $\mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$. The first unit diagram in the definition is an arbitrary component of the following triangle:

$$\begin{array}{ccc} \mathcal{C} \boxtimes \mathcal{I} & & \\ {\scriptstyle id \boxtimes \eta}\downarrow & \searrow^{\cong} & \\ \mathcal{C} \boxtimes \mathcal{C} & \xrightarrow[\mu]{} & \mathcal{C} \end{array},$$

and the second unit diagram in the definition is a component of

$$\begin{array}{ccc} \mathcal{I} \boxtimes \mathcal{C} & & \\ {\scriptstyle \eta \boxtimes id}\downarrow & \searrow^{\cong} & \\ \mathcal{C} \boxtimes \mathcal{C} & \xrightarrow[\mu]{} & \mathcal{C} \end{array},$$

whose commutativity implies the unit diagram in the definition of monoids in a monoidal category. Finally, the first equivariance diagram in the definition says that it indeed factors through the tensor product $\otimes_{\kappa[\Sigma_k]}$ over $\kappa[\Sigma_k]$, which is the coequalizer, and the second equivariance diagram says that it factors through the tensor product $\otimes_{\kappa[\prod_{r=1}^{k}\Sigma_{j_r}]}$ over $\kappa[\prod_{r=1}^{k}\Sigma_{j_r}]$. □

With this said, we can define the associated monad of an operad. But we first define the associated endofunctor of a Σ-object.

Definition 7. *Define a functor* $\mathrm{C} \in \mathfrak{S}^{\mathfrak{S}} = End_{\mathfrak{Cat}}(\mathfrak{S})$ *associated to a* Σ*-object* $\mathcal{C}$ *by*

$$\mathrm{C}x = \coprod_{j\geq 0} \mathcal{C}(j) \otimes_{\kappa[\Sigma_j]} x^{\otimes j},$$

where $\mathcal{C}(0) \otimes_{\kappa[\Sigma_0]} x^0 = \kappa$.

Theorem 6. *The functor*

$$(\mathfrak{S}^{\Sigma^{op}}, \boxtimes, \mathcal{I}) \to (\mathfrak{S}^{\mathfrak{S}}, \circ, \mathrm{Id})$$

specified by the above definition is a (strong) monoidal functor, thus it induces a functor

$$\mathbf{Mon}(\mathfrak{S}^{\Sigma^{op}}, \boxtimes, \mathcal{I}) \to \mathbf{Mon}(\mathfrak{S}^{\mathfrak{S}}, \circ, \mathrm{Id})$$

on monoids, which means it sends operads in $\mathfrak{S}$ *to monads in* $\mathfrak{S}$.

Proof. Omitted. □

Theorem 7. *Let* $\mathcal{C}$ *be an operad and* C *its associated monad in* $\mathfrak{S}$*. Then the category* $\mathcal{C}$-$\mathfrak{Alg}$ *of algebras over* $\mathcal{C}$ *is naturally isomorphic to the category* C-$\mathfrak{Alg}$ *of algebras over the monad* C*, natural in* $\mathcal{C}$.

Proof. The proof is straight forward. The associativity and unit diagrams in Definition 3 correspond to the associativity and unit diagrams in Definition 5; the equivariance diagram says that the action factors through the tensor product $\otimes_{\kappa[\Sigma_j]}$ over $\kappa[\Sigma_j]$. □

The following is a direct consequence of Proposition 1 and Theorem 6.

Proposition 8. *Let* $\mathcal{C}$ *be an operad in* $\mathfrak{S}$*. Then the forgetful functor* $U : \mathcal{C}$-$\mathfrak{Alg} \to \mathfrak{S}$ *sending* $(A, \theta : \mathcal{C}(j) \otimes A^j \to A)$ *to* A *has* $\mathrm{C} : \mathfrak{S} \to \mathcal{C}$-$\mathfrak{Alg}$ *as its left adjoint functor, where*

$$\mathrm{C}x = \coprod_{j\geq 0} \mathcal{C}(j) \otimes_{\kappa[\Sigma_j]} x^{\otimes j},$$

with the natural $\mathcal{C}$*-algebra structure on it.*

One application of operads is to describe algebraic types, to store different algebraic types into different operads in such a way that the category of this type of algebras is isomorphic to the category of algebras over this operad. Next we will give some examples on how classical algebraic types are hidden into the corresponding operads.

Example 2. Set $\mathcal{M}(j) = \kappa[\Sigma_j]$. The right action on $\mathcal{M}(j)$ is given by the right regular representation of Σ_j. The map $\eta : \kappa \to \mathcal{M}(1) = \kappa[\Sigma_1] = \kappa$ is the identity. The maps γ are induced by the maps

$$\Sigma_k \times \prod_{r=1}^{k} \Sigma_{j_r} \to \Sigma_j$$

defined by $\langle \sigma, \tau_{j_1}, \cdots, \tau_{j_k}\rangle \mapsto \sigma(j_1, \cdots, j_k) \circ \bigoplus_{r=1}^{k} \tau_{j_r}$.

Lemma 9. *The category* $\mathbf{Mon}(\mathfrak{S}, \otimes, \kappa)$ *of monoids in* $\mathfrak{S}$ *is isomorphic to the category* $\mathcal{M}$-$\mathfrak{Alg}$ *of algebras over* $\mathcal{M}$.

Proof. Assume $(A, \mu, \eta) \in \mathbf{Mon}(\mathfrak{S}, \otimes, \kappa)$. We will show that there is a natural $\mathcal{M}$-algebra structure on A. Let $b : A^2 \to A^2$ be the braiding map. Since the tensor product is distributive over the coproducts, $\kappa[\Sigma_j] \otimes A^j \cong \coprod_{\Sigma_j} A^j$ via the natural morphism. We define $\theta_0 : \coprod_1 A^0 = \kappa \to A$ to be $\eta : \kappa \to A$. Let $\gamma_{a,b} : a \otimes b \to b \otimes a$ be the natural braiding in $\mathfrak{S}$, and let $\gamma_{\sigma;a_1,\cdots,a_j} : \bigotimes_{r=1}^j a_r \to \bigotimes_{r=1}^j a_{\sigma r}$ be the iterated braiding as indicated in the indices, where $\sigma \in \Sigma_j$. This map is independent of the choice of orders of braidings, because in a symmetric monoidal category coherence theorem holds to ensure that different choices of orders with the same source and target all give the same iterated braiding. Then we define $\theta_j : \kappa[\Sigma_j] \otimes A^j \cong \coprod_{\Sigma_j} A^j \to A$ by setting the component of $\sigma \in \Sigma_j$ to be

$$\gamma_{\sigma;A,\cdots,A} : A^{\otimes j} \to A^{\otimes j}$$

followed by the iterated product map

$$\mu \circ (id_A \otimes \mu) \circ \cdots \circ (id_{A^{j-2}} \otimes \mu) : A^j \to A^{j-1} \to \cdots \to A$$

induced by $\mu : A^2 \to A$. One checks that with this action θ, we get an algebra over $\mathcal{M}$. Conversely, given an algebra over $\mathcal{M}$, say,

$$\theta_j : \kappa[\Sigma_j] \otimes A^j \cong \coprod_{\Sigma_j} A^j \to A$$

for each $j \geq 0$, we get a monoid structure on A by setting $\eta : \kappa \to A$ to be $\theta_0 : \coprod_1 A^0 \cong \kappa \to A$ and $\mu : A^2 \to A$ to be

$$\iota_e : A^2 \rightarrowtail \coprod_{\Sigma_2} A^2$$

followed by

$$\theta_2 : \coprod_{\Sigma_2} A^2 \to A.$$

One checks immediately that these two maps μ and η makes A into a monoid in $\mathfrak{S}$. Moreover, these two processes are inverse to one another. □

From a monoid, all the j-ary operations on it are iterated products with different orders, which are exactly what we "hide" in the operad $\mathcal{M}$, hence the algebras over $\mathcal{M}$ are just monoids, as expected.

One can also show that if A is an algebra over $\mathcal{M}$, then an A-module in the operadic sense is an A-bimodule in the classical sense of commuting left and right actions $A \otimes M \to M$ and $M \otimes A \to M$.

Example 3. Set $\mathcal{N}(j) = \kappa$ for $j \geq 0$, with trivial right action by Σ_j. The maps η and γ in the definition are all canonical isomorphisms.

Lemma 10. *The category of commutative monoids in* $\mathfrak{S}$ *is isomorphic to the category* $\mathcal{N} - \mathfrak{Alg}$ *of algebras over* $\mathcal{N}$.

Proof. Similar to the previous lemma. □

Since the multiplication is now commutative, the only j-ary operation on a commutative monoid is the iterated product in any order, which is what we put into the operad $\mathcal{N}$.

By the way, now a unital operad can be defined to be an operad over $\mathcal{N}$, i.e., an operad morphism $\varepsilon : \mathcal{C} \to \mathcal{N}$.

There exists the so-called "free operad" over any set of j-ary operators for $j \geq 0$, and the so-called "quotient operad" divided by a set of relations among the operators therein. So like any classical algebra, an operad can be presented by a set of generators and relations. A practical way to store an algebraic type into an operad is to find all operations in the algebraic type to form a free operad, and then find all relations in the axioms of the algebraic type to divide out. Since classical algebras all have explicit "j to 1" operations as well as relations, which have the shape of trees, operads derived this way are usually called "tree operads".

Example4**.** The Lie operad $\mathfrak{Lie}$ gives a typical example of a tree operad. To begin, by a j-tree we always mean a connected planar tree which has one root edge and j labelled (ordered) leaves indexed by integers 1 through j, and each vertex has valence ≥ 3, i.e., it must have ≥ 2 inputs at each vertex. Two j-trees are said to be equivalent if they are isomorphic as trees with ordered leaves, that is, if they are "identical". Now our ground category is $(\mathfrak{Mod}(k), \otimes_k, k)$, where k is the ground field of characteristic other than 2. Consider the vector space $\mathbf{V}(j)$ over k with basis the equivalence classes of **binary** j-trees, i.e., each vertex has valence 3. For example $\mathbf{V}(0) = 0$, since there's no such trees; $\mathbf{V}(1) = k$, since there's only one such tree. There are also the "grafting maps"

$$\gamma_{l;j_1,\cdots,j_l} : \mathbf{V}(l) \otimes \bigotimes_{r=1}^{l} \mathbf{V}(j_r) \to \mathbf{V}(\sum_r j_r)$$

defined by grafting those j_r-trees onto a l-tree via the ordering on the leaves of the l-tree to form a $(\sum_r j_r)$-tree. Then a subspace $W_1 \subset \mathbf{V}(j_1)$ generates a subspace $W \subset \mathbf{V}(j)$ for any fixed $j \geq j_1$ as follows:

$$W = span\{ \bigcup_{l, j_2, \cdots, j_l, \ s.t. \ \sum_{r=1}^{l} j_r = j} \gamma_{l;j_1,\cdots,j_l}(\mathbf{V}(l) \otimes W \otimes \bigotimes_{r=2}^{l} \mathbf{V}(j_r))\}.$$

Denote it by $W = W(W_1 \subset \mathbf{V}(j_1); \mathbf{V}(j))$. Now we take the subspace $W_1 \subset \mathbf{V}(2)$ to be the one generated by the vector:

and take $W_2 \subset \mathbf{V}(3)$ to be the subspace generated by the vector:

We set $\mathfrak{Lie}(j)$ for $j \geq 0$ to be the quotient space of $\mathbf{V}(j)$ divided out by the subspace $\sum_{i=1,2, \ and \ i+1 \leq \ j} W(W_i \subset \mathbf{V}(i+1); \mathbf{V}(j))$. The k-space $\mathfrak{Lie}(j)$ admits an action by Σ_j by permuting the ordering of the leaves. The grafting maps $\gamma_{l;j_1,\cdots,j_l}$ pass to quotients to give:

$$\gamma_{l;j_1,\cdots,j_l} : \mathfrak{Lie}(l) \otimes \bigotimes_{r=1}^{l} \mathfrak{Lie}(j_r) \to \mathfrak{Lie}(j),$$

where $j = \sum_r j_r$. Since $\mathfrak{Lie}(1) \cong k$, generated by the single 1-tree, we define $\eta : k \to \mathfrak{Lie}(1)$ to be the identity map, that is, it maps $1 \in k$ to this 1-tree. One sees easily that the composition maps γ is compatible with the actions by symmetric groups in the sense of the equivariance condition in the definition of an operad. So $\mathfrak{Lie}$ is an operad in $(\mathfrak{Mod}(k), \otimes_k, k)$.

Lemma 11. *The category of Lie algebras over the field k with char$(k) \neq 2$ is isomorphic to the category of algebras over the operad $\mathfrak{Lie}$. In particular, if* Lie *is the associated monad on $\mathfrak{Mod}(k)$, then for any $V \in \mathfrak{Mod}(k)$, the enveloping algebra of V is isomorphic to* Lie(V) *naturally.*

Proof. It's a routine matter to check. □

Example 5. Another such example is the Poisson operad.

Definition 8. *A Poisson algebra over a field k of characteristic* 0 *is a vector space V over k with three maps:*

$$e : k \to V, \ \cdot : V \otimes V \to V, \ [\ ,\ ] : V \otimes V \to V,$$

such that e and $\cdot$ makes V into a commutative monoid, $[\ ,\ ]$ makes V into a Lie algebra, and the bracket function $[a, *]$ is a derivation for the dot product $\cdot$ in the sense that

$$[a, b \cdot c] = [a, b] \cdot c + b \cdot [a, c]$$

for all $a, b,$ and $c \in V$.

To construct the so-called "Poisson operad", we need two k-linearly independent binary operations in $\mathfrak{Poisson}(2)$, one for the dot product $\cdot$ and one for the bracket $[\ ,\ ]$. This means that we will have two different vertices on a tree, say colored red and black. Also there is now a unit for one binary operation, the dot product $\cdot$, so we have to get a 0-red tree.

Definition 9. *The generators consists of a* 0*-red tree, a* 2*-red tree and a* 2*-black tree. Like the Lie operad, we define $\mathbf{V}(j)$ to be the k-vector space generated by the equivalence classes of connected planar binary j-trees with each vertex colored red or black. Note that now we include a* 0*-red tree. We also have the grafting maps*

$$\gamma_{l;j_1,\cdots,j_l} : \mathbf{V}(l) \otimes \bigotimes_{r=1}^{l} \mathbf{V}(j_r) \to \mathbf{V}(\sum_r j_r).$$

Now we have a "larger" subspace to divide out. Let $W_1 \subset \mathbf{V}(1)$ be the subspace generated by the following two vectors:

let $W_2 \subset \mathbf{V}(2)$ be the subspace generated by the vector

finally let $W_3 \subset \mathbf{V}(3)$ be the subspace generated by the following vectors:

Again let

$$W(W_1 \subset \mathbf{V}(j_1); \mathbf{V}(j)) = span\{ \bigcup_{l, j_2, \cdots, j_l,\ s.t.\ \sum_{r=1}^{l} j_r = j} \gamma_{l; j_1, \cdots, j_l}(\mathbf{V}(l) \otimes W \otimes \bigotimes_{r=2}^{l} \mathbf{V}(j_r))\},$$

and define $\mathfrak{Poisson}(j)$ to be the quotient space of $\mathbf{V}(j)$ modulo the subspace

$$\sum_{i=1,2,3,\ and\ i \leq j} W(W_i \subset \mathbf{V}(i); \mathbf{V}(j))$$

for $j \geq 0$. The grafting maps pass to quotients to give:

$$\gamma_{l; j_1, \cdots, j_l} : \mathfrak{Poisson}(l) \otimes \bigotimes_{r=1}^{l} \mathfrak{Poisson}(j_r) \rightarrow \mathfrak{Poisson}(j),$$

which is compatible with the actions by symmetric groups. We call this operad in $\mathfrak{Mod}(k)$ the Poisson operad over k.

Lemma 12. *The category of Poisson algebras is isomorphic to the category of algebras over the Poisson operad.*

Proof. Routine. □

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