4.9.

X can be embedded into some $\mathbb{P}^n \times Y$, and Y can be embedded into some $\mathbb{P}^m \times Z$, so X can be embedded into $\mathbb{P}^n \times \mathbb{P}^m \times Z$. So all we need to show is that the Segre embedding $\mathbb{P}^n_{\mathbb{Z}} \times \mathbb{P}^m_{\mathbb{Z}} \to \mathbb{P}^{nm+n+m}_{\mathbb{Z}}$ is a closed embedding. One possible way to do is to cover the target $\mathbb{P}^{nm+n+m}$ by open affines.

4.10.

For (a), let $\{X_i\}$ be irreducible components of X, and suppose we have Chow's lemma for irreducible schemes. So let $X_i' \to X_i$ and $U_i \subset X_i$ satisfy the assumption. Now we can take U_i to be smaller in order for U_i's to be disjoint (e.g., take the intersection of U_i with $X_i - \bigcup_{j \neq i} X_j$). Then show $X' := \coprod_i X_i'$ is also projective over S, and $X' \to X$ and $U := \coprod_i U_i$ satisfy the required property.

For (b), consider first the case where $S = \text{Spec } A$ is affine. Take any open affine $U \subset X$. Then $U \cong \text{Spec } A[X_1, \cdots, X_r]/I$ for some ideal I. In classical algebraic geometry given an embedded affine variety $Y \subset \mathbb{A}^r$ we can take its "projective closure", which is the closure of Y in $Y \subset \mathbb{A}^r \subset \mathbb{P}^r$, by homogenelize a set of generators of I. So do the same thing to U to show it's quasi-projective.

5.1.

We can cover X by open sets $\{U_i\}$ over each of which $\mathscr{E}$ is free. Fix an isomorphism $\varphi_i : \mathscr{E}|_{U_i} \xrightarrow{\sim} \mathscr{O}^r_{U_i}$. Note that the two sides (for any one of (a), (b), (c), (d)) are trivially isomorphic if $\mathscr{E}$ is free. Then define an isomorphism between the two sides restricted to U_i, using φ_i, and show they agree on overlaps. The point is that if one defines the local isomorphisms correctly, when going from one side to the other one uses both φ_i and φ_i^{-1}, and they sort of "cancel" so that the local isomorphism is base-free, i.e. independent of the φ_i's chosen, so they glue.

Remark: if $\mathscr{E}$ is locally free of rank r, then $f^*\mathscr{E}$ is also locally free of rank r, because $f^*\mathscr{O}_Y = \mathscr{O}_X$.

For those who want to use Yoneda to do (d), the following two facts might be helpful. Let $\mathscr{E}$ be locally free of finite rank.

①. $\text{Hom}(\mathscr{F}, \mathscr{G} \otimes \mathscr{E}) = \text{Hom}(\mathscr{F} \otimes \check{\mathscr{E}}, \mathscr{G})$. This follows from (b) and (c).

②. $f^*(\mathscr{F} \otimes \mathscr{E}) = f^*\mathscr{F} \otimes f^*\mathscr{E}$. This can be proved for instance using the same argument as above.