1. you may want to prove the following general version of Yoneda's lemma, and use it to do the homework problem.

Lemma 1. *Let $\mathcal{C}$ be a category and a an object of $\mathcal{C}$. Let $F : \mathcal{C}^{op} \to Set$ be a functor. Then there is a natural bijection*

$$Nat(h_a, F) \overset{\simeq}{\to} F(a)$$

of sets, where $Nat(G, F)$ is the set of natural transformations from the functor G to the functor F.

The idea is this. Given a natural transformation $\varphi : h_a \to F$, applying it to a we get $h_a(a) = \mathrm{Hom}(a, a) \to F(a)$, and the image of the identity 1_a gives an element of the set $F(a)$. Conversely given some $x \in F(a)$, use the naturality to construct a natural transformation $h_a \to F$.

After this, you can take $F = h_b$ for some other object b in $\mathcal{C}$.

2. similar to the one in class.

3. the fiber products, or sometimes called pull-backs, in the dual category $\mathcal{C}^{\mathrm{op}}$ are called push-outs. The push-out in Set, of $Y \overset{g}{\leftarrow} Z \overset{f}{\to} X$ is $X \bigsqcup Y/\sim$ where $\sim$ is the equivalence relation generated by $(f(z), g(z))$ for all $z \in Z$, on the disjoint union $X \bigsqcup Y$. For (iii), construct the fiber product first in Set, and then give it a natural ring structure, and show it is the fiber product in Ring. For (iv), it is given by tensor product.

4. Some corrections first: $I \subset B$ is an ideal of B, and since I may not be an A-module, $I \otimes_A C$ makes no sense. So the extension I^e in this case is defined to be the ideal in $B \otimes_A C$ generated by the image of I under the natural map $b \mapsto b \otimes 1 : B \to B \otimes_A C$. Also it's not clear to someone that in which category are we asked to show they are isomorphic, like isomorphic as abelian groups ($\mathbb{Z}$-modules), rings ($\mathbb{Z}$-algebras), A-modules or A-algebras, etc. I guess it should be understood that they are isomorphic as algebras (so at least as rings), in order to apply ex.3(iv). In fact they are even isomorphic as $B \otimes_A C$-algebras. Let's just do it in Ring.

Note that in ex.3(iv) we know that the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "b \mapsto b \otimes 1"] \\
C \arrow[r, "c \mapsto 1 \otimes c"'] & B \otimes_A C
\end{tikzcd}$$

represents the functor Ring $\to$ Set sending a ring D to the set of commutative diagrams

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "u"] \\
C \arrow[r, "v"'] & D.
\end{tikzcd}$$

Explicitly, given u, v as above, the corresponding ring homomorphism $B \otimes_A C \to D$ is given by $b \otimes c \mapsto u(b)v(c)$.

Yoneda's lemma says if c is an object in a category $\mathcal{C}$, then c is determined up to unique isomorphism by either $h_c := \mathrm{Hom}_{\mathcal{C}}(-, c) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ or $h'_c := \mathrm{Hom}_{\mathcal{C}}(c, -) : \mathcal{C} \to \mathrm{Set}$. Here in order to use the universal property of tensor product as a pushout (or, in general, colimit) we choose to use h'_c, namely we will show the two functors $\mathrm{Hom}_{\mathrm{Ring}}((B/I) \otimes_A C, -)$ and $\mathrm{Hom}_{\mathrm{Ring}}((B \otimes_A C)/I, -) : \mathrm{Ring} \to \mathrm{Set}$ are isomorphic.

For any ring D, by the universal property, to give a ring homomorphism $(B/I) \otimes_A C \to D$ is the same as giving a commutative diagram in Ring:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B/I \arrow[d] \\ C \arrow[r] & D, \end{tikzcd}$$

and composing $B \to B/I \to D$ we see that this is the same as giving a commutative diagram in Ring:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "u"] \\ C \arrow[r, "v"'] & D, \end{tikzcd}$$

such that $u(I) = 0$. This is equivalent to giving a ring homomorphism $B \otimes_A C \xrightarrow{\alpha} D$ with some additional condition corresponding to $u(I) = 0$. If this condition is $\alpha(I^e) = 0$, then we are done, since to give such a ring homomorphism α is the same as giving a ring homomorphism $(B \otimes_A C)/I^e \to D$, and therefore those two hom functors are isomorphic. $\alpha(b \otimes c) = u(b)v(c)$, so if we compose $B \xrightarrow{b \mapsto b \otimes 1} B \otimes_A C \xrightarrow{\alpha} D$ we recover $u : B \to D$. If $\alpha(I^e) = 0$, then $u(I) \subset \alpha(I^e)$, so $u(I) = 0$. Conversely, if $u(I) = 0$, then $\alpha(i \otimes 1) = 0$ for any $i \in I$, and since I^e is the ideal generated by these tensors, $\alpha(I^e) = 0$, too. Note that $I \otimes_A C$ makes no sense, but $i \otimes_A 1$ makes perfect sense.

Some people want to use the tensor-hom adjunction, but that works only in the category of A-modules. Precisely, if M, N, P are A-modules, then we have a natural isomorphism of A-modules

$$\mathrm{Hom}_A(M \otimes_A N, P) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)),$$

where by Hom_A I mean $\mathrm{Hom}_{A\text{-Mod}}$. Also $\mathrm{Hom}_A(M \otimes_A N, P)$ is in natural bijection with the set of A-bilinear maps $M \times N \to P$. These won't be true if one replace A-Mod by A-Alg.

5. first construct the inverse limit in Set, and then give it a natural group structure.