

recommended reading: commutative algebra, by Atiyah-Macdonald, chapters 3 (on localization) and 5 (on integral ring extensions and valuation rings), and a later chapter (forget the number) on discrete valuation rings.

Let A be a commutative ring and $S \subset A$ a multiplicatively closed set, then we can form the ring of fractions $S^{-1}A$, called the localization of A at S . One can think of this ring as adjoining to A the inverses of elements in S , which may not have inverses in A . And prime ideals in $S^{-1}A$ correspond one-to-one to primes in A that are disjoint from S , inclusion-preserving. Let $\mathfrak{a} \triangleleft A$ be an ideal. Then $A - \mathfrak{a}$ is multiplicatively closed if and only if $\mathfrak{a} = \mathfrak{p}$ is a prime ideal. Write $A_{\mathfrak{p}}$ for $(A - \mathfrak{p})^{-1}A$, and call it the localization of A at \mathfrak{p} (is it confusing, rather than saying “at $S = A - \mathfrak{p}$ ”? sorry..) So prime ideals in $A_{\mathfrak{p}}$ correspond to prime ideals in A which are CONTAINED IN \mathfrak{p} . By the way, for a general ideal $\mathfrak{a} \triangleleft A$, (prime) ideals in A/\mathfrak{a} correspond to (prime) ideals in A CONTAINING \mathfrak{a} . So if $\mathfrak{a} \subset \mathfrak{p}$, then prime ideals in $(A/\mathfrak{a})_{\mathfrak{p}}$ are in one-to-one correspondence with prime ideals in A lying between \mathfrak{a} and \mathfrak{p} .

$A_{\mathfrak{p}}$ is a local ring with the unique maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$. That justifies the terminology: $S^{-1}A$ is called localization because in the special case where $S = A - \mathfrak{p}$, after we localize, we get a local ring. Well, you may ask why this algebraic construction is called localization, which sounds like something geometric, and why a ring with a unique maximal ideal (which is again something purely algebraic), is named to be local? What does this have to do with the geometry that someone sits there, staring at a point, or the region around it, among a whole picture, which is reasonable to be called “local”? i will explain this in later examples.

Keep on doing algebra. If A is an integral domain, then 0 is a prime ideal in A , and we define the field of fractions of A , also called fraction field or quotient field of A , denoted $K(A)$, to be the localization of A at 0 . Note that this process has nothing to do with taking quotient, or modding out by something. A sits in $K(A)$ as a subring. For instance, $K(\mathbb{Z}) = \mathbb{Q}$, $K(k[X]) = k(X)$, X being an indeterminate.

If (A, \mathfrak{m}) is a local ring, then call $k(A) := A/\mathfrak{m}$ the residue field of the local ring A . If A is a general ring, not necessarily to be local, and \mathfrak{p} is a prime ideal in A , then the residue field at \mathfrak{p} is defined to be the residue field of the local ring $A_{\mathfrak{p}}$, which is $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Recall that in general the functor

$$S^{-1} : \text{Mod}(A) \rightarrow \text{Mod}(S^{-1}A)$$

is exact, so it preserves all finite limits and finite colimits; in particular, it preserves kernels and cokernels. So $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = (A/\mathfrak{p})_{\mathfrak{p}A_{\mathfrak{p}}}$. Note that since \mathfrak{p} is prime, A/\mathfrak{p} is an integral domain, and \mathfrak{p}^e is the zero ideal in A/\mathfrak{p} , so what we get is that the residue field at \mathfrak{p} is also equal to the fraction field of the integral domain A/\mathfrak{p} .

Examples:

1. Number fields. Let $A = \mathbb{Z}$ and $\mathfrak{p} = (p)$, where p is a prime number. Then $\mathbb{Z}_{(p)} = \{\frac{a}{b} | a, b \in \mathbb{Z}, p \nmid b\}$ is a local domain whose fraction field is \mathbb{Q} and residue field is \mathbb{F}_p . If q is a power of p , can you find a local ring whose residue field is \mathbb{F}_q ? May want to go to number fields.

2. Function fields. Let k be a field. A function field K of degree n over k is a finitely generated field extension of k of transcendence degree n . You may want to give the definition of the transcendence degree of a field extension, and i will just give an example: the transcendence degree of $K(k[X, Y]/(X^2 + Y^2 - 1))$ over k is 1, since x and y (images of X and Y in the quotient ring) are algebraically dependent, so the freedom is 1. Note that the “dimension” of the circle $X^2 + Y^2 = 1$ is also supposed to be 1. This is not a coincidence, as we will find out later. These two numbers are always equal.

Let X be an integral scheme of finite type over k (or a variety over k , in the sense of Hartshorne II.4). Then X is irreducible, so there’s a unique generic point $\xi \in X$ such that

$\overline{\{\xi\}} = X$. Then in an exercise in Hartshorne II.3 one shows that the stalk $\mathcal{O}_{\xi, X}$ is a field, called the function field of X , denoted $k(X)$. We keep track of the base field k by writing the function field of X as $k(X)$ rather than $K(X)$. umm, $\mathcal{O}_{x, X} = \mathcal{O}_{X, x}$, so either way you like.

How the function field of a variety is defined in classical algebraic geometry, before schemes? Here is the way to define it. Let X be a (classical) variety over an algebraically closed field k , namely, not a scheme, but the k -points of that scheme with the Zariski topology and the pullback of the structure sheaf from the scheme. In particular, every point in X is a closed point. Let $x \in X$. Define $\mathcal{O}_{x, X}$, as a set, to be the quotient set of $\{(U, f) \mid x \in U \subset X \text{ open nbhd of } x, f : U \rightarrow k \text{ a regular function}\}$ under the equivalence relation that $(U, f) \sim (V, g)$ if $f|_{U \cap V} = g|_{U \cap V}$. A function $f : U \rightarrow k$ is called regular if locally it's given by polynomial equations. Think of holomorphic functions, which are locally given by convergent Taylor series. This quotient set has a natural ring structure, given as follows. Denote by $\langle U, f \rangle$ the image of the pair (U, f) in the quotient set. Define addition $\langle U, f \rangle + \langle V, g \rangle := \langle U \cap V, f|_{U \cap V} + g|_{U \cap V} \rangle$, and also multiplication in a similar way. Check this is well-defined and makes it a ring. Call $\mathcal{O}_{x, X}$ the local ring at $x \in X$. It is a local ring with maximal ideal $\mathfrak{m}_x = \{(U, f) \mid f(x) = 0\}$ and residue field $\mathcal{O}_x / \mathfrak{m}_x = k$.

Also define the function field $k(X)$ of X over k , as a set, to be the quotient set

$$\{(U, f) \mid \emptyset \neq U \subset X \text{ open, } f : U \rightarrow k \text{ regular}\} / \sim$$

where $(U, f) \sim (V, g)$ if the same condition holds as above. You may want to define addition and multiplication on this set in the same as we did above, and show by yourself that this makes it not only a ring but a field. So can think of elements in the function field of X as k -valued functions only defined (or regular) on an open subset of X , rather than whole of X . Define the (Krull) dimension of X . (see Hartshorne I.1) Say $\dim(X) = n$. Then we claim that the function field $k(X)$ of X is a function field of degree n over k , in the sense we defined above, namely, a finitely generated field extension over k with transcendence degree n . This follows from the following theorem in Hartshorne I without proof.

Theorem 1. *Let A be an integral domain, which is a finitely generated k -algebra, for some field k . Then $\dim(A) = \text{tr.deg. } K(A)/k$. Note that $\dim(A)$ here is the Krull dimension of the ring A , not the k -dimension $\dim_k A$ of the vector space A . And $K(A)$ is the fraction field of A .*

This is a deep theorem.

Now we defined $\mathcal{O}_{x, X}$ and $k(X)$. There's a more generalized notion, namely the local ring $\mathcal{O}_{Y, X}$ along a subvariety $Y \subset X$. Remember that by a variety we mean an irreducible closed subset, with the reduced induced sheaf structure. You can find this in an exercise in Hartshorne I,3, and i will sketch the definition here. It's defined exactly in the same way as above, where in (U, f) we require $U \cap Y \neq \emptyset$ so that f can be defined on a non-empty open subset of Y , namely $U \cap Y$. Some properties of it: it's again a local ring, with fraction field $k(X)$ and residue field $k(Y)$. Let me repeat: its residue field is the FUNCTION FIELD of Y . And its dimension $\dim(\mathcal{O}_{Y, X})$ is the codimension $\text{codim}(Y, X)$ of Y in X . So in particular, if $\text{codim}(Y, X) = 1$, then $\dim(\mathcal{O}_{Y, X}) = 1$, which is what we did in today's lecture. I will say more later after we introduce valuation rings. If we take $Y = x$ to be a (closed) point, then $\mathcal{O}_{Y, X} = \mathcal{O}_{x, X}$, of dimension $\text{codim}(x, X) = \dim(X)$. If we take $Y = X$, then $\mathcal{O}_{Y, X} = k(X)$ of dimension $\text{codim}(X, X) = 0$, which is it is supposed to be because a field always has dimension 0.

In the scheme setting, let X be a variety over k in the sense of Hart. II.4, then any subvariety $Y \subset X$ has a unique generic point $\eta_Y \in Y$ in it, with the property that the closure of $\{\eta_Y\}$ in X is equal to Y . Then the local ring $\mathcal{O}_{Y, X}$ along Y we defined above

is nothing else but $\mathcal{O}_{\eta_Y, X}$, the stalk of the structure sheaf \mathcal{O}_X at the point η_Y . Again have $\dim(\mathcal{O}_{\eta_Y, X}) = \text{codim}(Y, X)$. Can also write it as $\dim(\mathcal{O}_{x, X}) = \text{codim}(\overline{\{x\}}, X)$, for any scheme point $x \in X$, not necessarily closed.

Now turn to valuation rings, especially discrete valuation rings (or DVR).

Definition 1. A pair (G, \geq) is called a totally ordered abelian group if G is an abelian group with a total order \geq on its underlying set (a total order is a partial order such that one can compare any two elements), and the two structures are compatible in the sense that $x \geq y \Rightarrow x + z \geq y + z$.

Examples: \mathbb{Z} , \mathbb{R} , ...

Definition 2. Let K be a field and (G, \geq) a totally ordered abelian group. A (G, \geq) -valuation, or just a G -valuation, on K is a homomorphism of groups $v : K^\times \rightarrow G$ such that $v(x + y) \geq \min(v(x), v(y))$. For conveniens we usually set $v(0) = \infty$. A (\mathbb{Z}, \geq) -valuation is usually called a discrete valuation. The valuation ring associated to the valuation v is defined to be $R_v := \{x \in K^\times | v(x) \geq 0\} \cup \{0\}$. Easy to show it's a subring of K . One can also show that R_v is an integrally closed domain, with fraction field K , and for any $x \in K^\times$, either $x \in R_v$ or $x^{-1} \in R_v$. An integral domain R is called a valuation ring if it's the valuation ring associated to some valuation on its fraction field $K(R)$.

I will try to list all basic properties a DVR has.

Theorem 2. Let R be a DVR. Then R is a 1-dimensional local integrally closed PID. Let \mathfrak{m} be the maximal ideal in R , say generated by π . Then every ideal in R has the form (π^n) for a uniquely determined $n \geq 0$. Every element in the fraction field K of R can be written as $u\pi^n$ for some unit $u \in R^\times$ and a uniquely determined $n \in \mathbb{Z}$, and R consists of those $u\pi^n$ where $n \geq 0$. If R is the valuation ring associated with the discrete valuation $v : K^\times \rightarrow \mathbb{Z}$, then $v(u\pi^n) = n$. Let $k = R/\mathfrak{m}$ be the residue field of R , then $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$.

Definition 3. Let (A, \mathfrak{m}) be a local ring of dimension d , and $k = A/\mathfrak{m}$ the residue field. Then A is a regular local ring if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = d$.

So a DVR is a 1-dimensional regular local ring. The converse is also true.

Examples of DVR's:

1. **number fields.** $\mathbb{Z}_{(p)}$. It's a local ring, and $\pi = p$. Let $v : \mathbb{Q}^\times \rightarrow \mathbb{Z}$ be the corresponding discrete valuation. Then $v(\frac{a}{b})$ is the power that p divides $\frac{a}{b}$. If $v(\frac{a}{b}) \geq 0$, namely $p \nmid b$, then by definition $\frac{a}{b} \in \mathbb{Z}_{(p)}$. So it is the valuation ring associated with this discrete valuation, and the residue field of it is \mathbb{F}_p .

More generally, let K be a number field (i.e., a finite extension of \mathbb{Q}) and \mathcal{O}_K the ring of integers in K . Let \mathfrak{p} be a non-zero prime ideal in \mathcal{O}_K , then the localization $(\mathcal{O}_K)_{\mathfrak{p}}$ is a DVR. If $\mathfrak{p} \cap \mathbb{Z} = (p)$, then the residue field of it is \mathbb{F}_q where $q = p^f$ is a power of p . f is called the residue degree.

Before giving the second example let's go back to classical algebraic geometry, considering varieties (not a scheme) defined over an algebraically closed field k , and define the non-singularity of a (closed) point on a variety quickly. This is an important notion. Let X/k be a variety and $x \in X$ a (closed) point. Define the local dimension $\dim_x X$ of X at x to be the Krull dimension $\dim(\mathcal{O}_{x, X})$, and there's a theorem (theorem 11.25 in Atiyah-Macdonald) which says $\dim_x X$ is always equal to $\dim(X)$, for any $x \in X$. Note that a variety is always assumed to be irreducible, so this is not surprising. The tangent space of X at x is defined to be $\text{Hom}_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x))$, the dual vector space of $\mathfrak{m}_x/\mathfrak{m}_x^2$. If you know before some other definitions of a tangent space in differential geometry, you should try to put them in natural isomorphisms. Then intuitively, if x is a "non-singular" point in X

(which we haven't defined), like in a manifold, then the local dimension at x should be the same as the dimension of the tangent space at x as a $k(x)$ -vector space. So we give the following intrinsic definition: a point $x \in X$ is non-singular if $\dim(\mathcal{O}_{x,X}) = \dim_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2)$. This is equivalent to saying that $\mathcal{O}_{x,X}$ is a regular local ring. Here we used the cotangent space $\mathfrak{m}_x/\mathfrak{m}_x^2$, which doesn't matter because everything is finite-dimensional, so they have the same dimension. X is non-singular if all points on it are non-singular.

2. **function fields.** $k[X]_{(f(X))}$, where $f(X) \in k[X]$ is an irreducible polynomial so that $(f(X))$ is a prime ideal. Then $\pi = f(X)$, and the discrete valuation corresponding to it is similar to the above: it's the power that $f(X)$ divides. The residue field is $k[X]/(f(X))$. Note that in particular, if k is algebraically closed, then $f(X)$ must be of degree 1. Say $f(X) = X - a$ for some $a \in k$. What do you think the geometric picture of it should be? Recall that DVR's are 1-dimensional regular local rings, so if a DVR is a localization of some finitely generated k -algebra, then it can be regarded as the local ring of a curve at a non-singular point. Here $k[X]$ corresponds to \mathbb{A}_k^1 , which is non-singular everywhere, and so $k[X]_{(X-a)}$ is the local ring of $\mathbb{A}_k^1(k) = k$ at the point a . By the way, I think we already seen why a ring with a unique maximal ideal is called local: that's because a large class of such rings comes from $\mathcal{O}_{x,X}$, and by considering this ring one is really focusing at the point $x \in X$, hence the name "local".

Like in the case of number fields, if L is some finite separable extension of $k(X)$ and B is the integral closure of $k[X]$ in L . Then unlike $k[X]$, B may not be a PID. But it's still a Dedekind domain, which is by definition an integrally closed 1-dimensional noetherian domain. So let \mathfrak{p} be a non-zero prime (which is to say maximal) ideal in B , then $B_{\mathfrak{p}}$ is again a DVR. What's the geometry of that? Did you see the interesting analogy between number fields and function fields? There're much more we can say about this analogy, but not here.

Back to our lecture. If X satisfies $(*)$, in particular, X is regular in codimension 1, then for any integral closed subscheme Y in X of codimension 1, the local ring $\mathcal{O}_{\eta_Y, X}$ has dimension 1, and it's assumed to be regular, and also noetherian, since X is noetherian, this ring must be a DVR, with fraction field $k(X)$ and residue field $k(Y)$. Denote by v_Y the associated discrete valuation on $k(X)^*$. Then for any rational function $f \in k(X)^*$, we have an integer $v_Y(f)$. What is it? Think of the complex plane and a meromorphic function f on it. Then for each point p in \mathbb{C} we can talk about $v_p(f)$, the degree of the leading term of the Laurent expansion of f at p . So when f has a zero at p , $v_p(f) > 0$; when f has a pole at p , $v_p(f) < 0$. And this number is called the order of f at p , and there're some residue theorems about it. Now in higher dimensional case, the zero locus or pole of a rational function is no longer a finite collection of points, but a finite collection of some codimension 1 subvarieties. And since v_Y is still a discrete valuation, we can still talk about like the vanishing order of f along Y , or degree of f along the pole locus Y . This number is $v_Y(f)$.