1.

Proof. Hint: Recall that in the definition of a vector bundle over Y in (II, Ex.5.18), the "total scheme" X and the map $\pi : X \to Y$ are given. So here we have to reconstruct the scheme X and the map π. Use gluing of schemes (II, Ex.2.12). Rest is routine.

□

2.

Proof. Take the open cover $\mathbb{P}^n_k = \bigcup_i U_i$ where $U_i = D_+(X_i)$. Then $U_i =$Spec $k[\frac{X_0}{X_i}, \cdots, \frac{\hat{X_i}}{X_i}, \cdots, \frac{X_n}{X_i}]$ =Spec $k[\frac{X_j}{X_i}; j]$, and $U_i \cap U_j =$Spec $k[\frac{X_k X_l}{X_i X_j}; k, l]$. $\mathbb{A}^1_{U_i \cap U_j} =$Spec $k[\frac{X_k X_l}{X_i X_j}, T; k, l]$. Define $\theta_{i,j} : \mathbb{A}^1_{U_i \cap U_j} \to \mathbb{A}^1_{U_i \cap U_j}$ to be the morphism induced by $\phi_{i,j} : k[\frac{X_k X_l}{X_i X_j}, T; k, l] \to k[\frac{X_k X_l}{X_i X_j}, T; k, l]$ defined by $T \mapsto \frac{X_j}{X_i} T$ and extended $k[\frac{X_k X_l}{X_i X_j}; k, l]$-linearly. The ideal is to think of the first $\mathbb{A}^1_{U_i \cap U_j}$ as Spec $k[\frac{X_k X_l}{X_i X_j}, X_i; k, l]$, and think of the second $\mathbb{A}^1_{U_i \cap U_j}$ as Spec $k[\frac{X_k X_l}{X_i X_j}, X_j; k, l]$, and note that $X_j = \frac{X_j}{X_i} X_i$. Then $\theta_{i,j}$ is a morphism over $U_i \cap U_j$ and all three conditions in Ex.1 are satisfied.

□

3.

Proof. In all the following 4 parts we are going to cover X by open subsets U_i such that $\mathcal{E}|_{U_i}$ is free of rank say r on every U_i, and define an isomorphism from one to the other on each U_i, and show on $U_i \cap U_j$ these two maps agree so that we can glue these isomorphisms. Note that the rank may differ on different connected components, but we can assume X is connected to start with, and then on each U_i the sheaf $\mathcal{E}$ will have the same rank. In the following when i write "=" i mean a natural isomorphism, in contrast to an isomorphism depending on some other choices like a trivialization, which we will denote by $\cong$ or $\tilde{\to}$.

(a). Suppose on $U \subset X$ the sheaf $\mathcal{E}$ is free, and choose a trivialization $\varphi : \mathcal{E}|_U \cong \mathcal{O}|_U^r$. Then we reduce the problem to the case where (by taking $X = U$) $\mathcal{E}$ is free on X, and define an isomorphism $(\check{\mathcal{E}})\check{} \cong \mathcal{E}$, using a trivialization, but the isomorphism should be independent of the trivialization chosen (so that on $U_i \cap U_j$, on which we have two trivializations, the two isomorphisms agree). We want to define a map $\mathcal{H}om(\mathcal{H}om(\mathcal{E}, \mathcal{O}), \mathcal{O}) \to \mathcal{E}$, and show it is an isomorphism, and it is independent of the trivialization chosen. It should be clear that $\mathcal{H}om(\mathcal{O}, \mathcal{F}) = \mathcal{F}$ naturally, and therefore $\mathcal{H}om(\mathcal{O}^s, \mathcal{F}) = \mathcal{F}^s$ naturally. Let $\varphi : \mathcal{E} \tilde{\to} \mathcal{O}^r$ be an isomorphism. Then it induces an isomorphism $\mathcal{O}^r = \mathcal{H}om(\mathcal{O}^r, \mathcal{O}) \tilde{\to} \mathcal{H}om(\mathcal{E}, \mathcal{O})$, given by "composition" with φ. Then this map induces an isomorphism $\mathcal{H}om(\mathcal{H}om(\mathcal{E}, \mathcal{O}), \mathcal{O}) \to \mathcal{H}om(\mathcal{O}^r, \mathcal{O}) = \mathcal{O}^r$. Up to now in defining the isomorphism $(\check{\mathcal{E}})\check{} \to \mathcal{O}^r$ we only used the trivialization φ for once. Then use φ^{-1} to go from $\mathcal{O}^r$ back to $\mathcal{E}$. Now we have an isomorphism $(\check{\mathcal{E}})\check{} \tilde{\to} \mathcal{E}$, and since φ and φ^{-1} kind of cancel with each, making this isomorphism natural, i.e., independent of φ.

Another way to do this is to define maps in both directions abstractly, without using any trivialization so that they are automatically natural, and show they are inverse to each other.

(b). Again assume $\mathcal{E}$ is free and choose an isomorphism $\varphi : \mathcal{E} \tilde{\to} \mathcal{O}^r$. Then have $\mathcal{H}om(\mathcal{E}, \mathcal{F}) \tilde{\to} \mathcal{H}om(\mathcal{O}^r, \mathcal{F}) = \mathcal{F}^r = \mathcal{O}^r \otimes_{\mathcal{O}} \mathcal{F} \tilde{\to} \check{\mathcal{E}} \otimes_{\mathcal{O}} \mathcal{F}$, where the first map involves φ and the last map involves φ^{-1}, so the isomorphism is natural.

(c). Assume $\mathcal{E}$ is free and choose an isomorphism $\varphi : \mathcal{E} \tilde{\to} \mathcal{O}^r$, and use it to define a series of isomorphisms $\mathrm{Hom}_{\mathcal{O}}(\mathcal{E} \otimes \mathcal{F}, \mathcal{G}) \tilde{\to} \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}^r \otimes \mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}^r, \mathcal{G}) = \bigoplus_r \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}^r) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{O}^r \otimes \mathcal{G}) \tilde{\to} \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \check{\mathcal{E}} \otimes \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{H}om(\mathcal{E}, \mathcal{G}))$, where the first $\cong$ is induced by φ^{-1} and the second by φ, so the whole composite is a natural isomorphism. Here the trick is that in an abelian category, for instance $\mathcal{O}_X - \mathcal{M}od$, finite coproduct is the same as finite product (think of direct sum of 2 abelian groups); and $\mathcal{O}^r$ is naturally self-dual, i.e., there's a natural perfect pairing $\mathcal{O}^r \otimes \mathcal{O}^r \to \mathcal{O}$. Note that a locally free sheaf, or even a free sheaf $\mathcal{E}$, is not naturally self-dual in general, and the difference between $\mathcal{O}^r$ and a free sheaf $\mathcal{E}$, like A^n and a free A-module of rank n, is that A^n is a free A-module of rank n together with a specified basis on it, namely e_i, which defines a non-degenerate pairing on itself.

(d). Same as above. do it yourself.

□

4.

Proof. For one direction, on an open affine Spec A, have $\tilde{M}$ and use the fact that M is the cokernel of some A-linear map of free A-modules, by first taking a set of generators for M (for instance, all elements in M) and using this to define a surjection $\varphi : A^\alpha \twoheadrightarrow M$, and then doing the same to the kernel. If the sheaf is coherent and A is noetherian, then M is a noetherian module, so both M and $\ker(\varphi)$ are finitely generated and we can choose those free modules to have finite ranks.

For the other direction, use (II, 5.7) and the fact that being quasi-coherent, or coherent, is a local property.

□

5.

Proof. (thank Manuel Reyes, from whom i got the solution) (a). Replacing X by an open affine nbhd. of x and translate to commutative algebra, we need to show the following:

suppose A is a noetherian ring and M a finitely generated A-module, and there's a prime ideal $\mathfrak{p} \lhd A$ such that $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module. Then there's an $f \in A - \mathfrak{p}$ such that M_f is a free A_f-module.

Say M is generated by $m_1, \cdots, m_r$, and $\frac{x_1}{a_1}, \cdots, \frac{x_l}{a_l}$ is a basis for $M_\mathfrak{p}$ over $A_\mathfrak{p}$. By taking $a = \prod a_i$ we can assume all $a_i = a \in A - \mathfrak{p}$. So have $\frac{m_j}{1} = \sum_i \frac{y_{i,j}}{b_{i,j}} \cdot \frac{x_i}{a}$. Again by taking $b = \prod_{i,j} b_{i,j}$ we can assume all $b_{i,j} = b \in A - \mathfrak{p}$. Those equations over $A_\mathfrak{p}$ means $s_j(abm_j - \sum_i x_i y_{i,j}) = 0$ in M, for some $s_j \in A - \mathfrak{p}$. Let $s = \prod_j s_j$. Then taking $f = abs$ will do.

(b). Follows from (a).

(c). Suppose $\mathcal{F}$ is invertible, and we will show $\mathcal{F} \otimes \check{\mathcal{F}} \cong \mathcal{O}_X$. Have $\mathcal{F} \otimes \check{\mathcal{F}} = \mathcal{H}om(\mathcal{F}, \mathcal{F})$ by (II, Ex.5.1). Do the same trick, by assuming $\mathcal{F}$ is free of rank 1, and choose an isomorphism $\varphi : \mathcal{F} \to \mathcal{O}_X$. Then note that $\mathcal{H}om(-,-)$, like the ordinary $\mathrm{Hom}(-,-)$, is contravariant in the first variable and covariant in the second, so in defining the isomorphism $\mathcal{H}om(\mathcal{F}, \mathcal{F}) \cong \mathcal{H}om(\mathcal{O}, \mathcal{O})$, φ and φ^{-1} both show up and they cancel. And $\mathcal{H}om(\mathcal{O}, \mathcal{O}) = \mathcal{O}$ since $\mathcal{O}$ is self-dual, as explained before. We also need $\check{\mathcal{F}}$ to be coherent. Hint: show locally it's $\mathrm{Hom}_A(M, A)^\sim$ on an open affine $U \cong$Spec A, if $\mathcal{F}|_U \cong \tilde{M}$.

Suppose $\mathcal{G} \otimes \mathcal{F} \cong \mathcal{O}_X$ for some coherent sheaf $\mathcal{G}$. Restricted to an open affine Spec A, have $\mathcal{F} \cong \tilde{M}$ and $\mathcal{G} \cong \tilde{N}$ for some finitely generated A-modules M and N, since they are both assumed to be coherent. Have $\tilde{M} \otimes_{\tilde{A}} \tilde{N} = (M \otimes_A N)^\sim$, and so $\mathcal{G}_x \otimes_{\mathcal{O}_x} \mathcal{F}_x = (\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G})_x$, for any point $x \in X$. Reduce the problem to the following: $M \otimes_A N \cong A$, where $(A, \mathfrak{m})$ is a noetherian local ring and both M and N are finitely generated A-modules, then $M \cong N \cong A$.

First we show both M and N are generated by one element. By Nakayama's lemma, only need to show both $\bar{M} := M/\mathfrak{m}M$ and $\bar{N} := N/\mathfrak{m}N$ are 1-dimensional vector spaces over $k = A/\mathfrak{m}$. For that we only need to show $\bar{M} \otimes_k \bar{N} \cong k$. Note that $k \otimes_A k = k/\mathfrak{m}k = k/0 = k$. So $k = k \otimes_A A = k \otimes_A (M \otimes_A N) = k \otimes_A k \otimes_A M \otimes_A N = (M \otimes_A k) \otimes_A (N \otimes_A k) = \bar{M} \otimes_A \bar{N}$.

This is a priori an A-module, but since $\mathfrak{m}$ acts trivially on it, it's a k-module. And there's a natural k-linear map $\bar{M} \otimes_A \bar{N} \to \bar{M} \otimes_k \bar{N}$, which is surjective. The domain is 1-dimensional, so the target has dimension 0 or 1. If it's 0 then $\bar{M}$ or $\bar{N} = 0$, and hence M or $N = 0$, contradicting with $M \otimes_A N \cong A$ and Spec $A \neq \emptyset$. So $\dim_k \bar{M} \otimes_k \bar{N} = 1$.

Now have $M \cong A/\mathfrak{a},\ N \cong A/\mathfrak{b}$ for some ideals $\mathfrak{a}, \mathfrak{b} \lhd A$. Then $A \cong M \otimes_A N \cong \frac{A}{\mathfrak{a}} \otimes_A \frac{A}{\mathfrak{b}} = \frac{A}{\mathfrak{a}+\mathfrak{b}}$ as A-modules. By taking annihilators on both sides we get $\mathfrak{a} + \mathfrak{b} = 0$, hence $\mathfrak{a} = \mathfrak{b} = 0$ and therefore $M \cong N \cong A$.

□

6.

Proof. If $g \in S_{n>0}$, then $g^d \in S_{nd}$. So if $\mathfrak{p} \lhd S$ is a homogeneous prime ideal, then $S_+ \not\subset \mathfrak{p} \Rightarrow \exists g \in S_{n>0}$ such that $g \notin \mathfrak{p} \Rightarrow g^d \notin \mathfrak{p} \Rightarrow \mathfrak{p} \in D_+(g^d)$. So $X =$Proj $S = \bigcup_{f \in S_{nd}, n \geq 1} D_+(f)$, where the union is taken over all homogeneous elements of degrees divisible by d. Let $X^{(d)} :=$Proj $S^{(d)}$. Obviously, $X^{(d)} =$Proj $S^{(d)} = \bigcup_{f \in S_n^{(d)}, n \geq 1} D_+^{(d)}(f)$. $D_+(f) =$Spec $S_{(f)},\ D_+^{(d)}(f) =$Spec $S_{(f)}^{(d)}$. Define a map $S_{(f)} \to S_{(f)}^{(d)}$ sending $\frac{h}{f^m}$ to $\frac{h}{f^m}$, for $m \geq 0,\ h \in S_{mnd} = S_{mn}^{(d)}$. It's a ring homomorphism. It's surjective, because any element in $S_{(f)}^{(d)}$ has the form $\frac{h}{f^m}$ for some $m \geq 0$ and $h \in S_{mn}^{(d)}$. It's injective, because both $S_{(f)}$ and $S_{(f)}^{(d)}$ are subrings of S_f, so an element in the subring is 0 if and only if it's 0 in S_f. So it defines an isomorphism $D_+^{(d)}(f) \xrightarrow{\sim} D_+(f)$. Left to you to show they are compatible on intersections. So they glue to give an isomorphism $\rho : X^{(d)} \xrightarrow{\sim} X$. Now prove $\rho_* \mathcal{O}_{X^{(d)}}(1) \cong \mathcal{O}_X(d)$. $\mathcal{O}_{X^{(d)}}(1)|_{D_+^{(d)}(f)} = (S^{(d)}(1)_{(f)})^{\sim},\ \mathcal{O}_X(d)|_{D_+(f)} = (S(d)_{(f)})^{\sim}$. Define $S(d)_{(f)} \to S^{(d)}(1)_{(f)}$ by $\frac{h}{f^m} \mapsto \frac{h}{f^m}$, where $h \in S_{ndm+d} = S_{nm+1}^{(d)} = S(d)_{mnd} = S^{(d)}(1)_{nm}$. It's an isomorphism of $S_{(f)}$-modules. To glue these together, we need to prove they agree on intersections, which is again left to you. Just some commutative diagrams.

□