Math 55 Quiz 1 Solutions

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1. Determine whether $\forall x \exists y (P(x) \wedge Q(y))$ and $\forall y \exists x (P(x) \wedge Q(y))$ are logically equivalent. If so, prove it. If not, provide a counterexample. (5 points)

   These statements are not logically equivalent. For example, let the domain be all integers, $P(x)$ the statement "x is an integer", and $Q(x)$ the statement "$x = 1$". Then $\forall x \exists y (P(x) \wedge Q(y))$ is true because for any x, you can take y to be 1 and then $P(x) \wedge Q(y)$ is true. But $\forall y \exists x (P(x) \wedge Q(y))$ is false because for $y = 2$, there does not exist an x such that $(P(x) \wedge Q(y))$ since $Q(y)$ is false.

2. Let $P(x)$ be the statement "x is prime" and $Q(x)$ be the statement "x is divisible by 5". Determine the truth value of the following statements if the domain of each variable consists of positive integers. Note that 1 is not prime. (1 point each)

   - $\exists x (P(x) \wedge Q(x))$

     True. Take $x = 5$.

   - $\forall x (Q(x) \leftrightarrow Q(x + 5))$

     True. If x is divisible by 5 then $x = 5y$ for some y, and then $x + 5 = 5(y + 1)$ so $x + 5$ is divisible by 5. Conversely, if $x + 5$ is divisible by 5 then $x + 5 = 5y$ for some y, so $x = 5(y - 1)$ is also divisible by 5.

   - $\forall x (P(x) \to \neg P(x - 1))$

     False. The number 3 is prime, and $3 - 1 = 2$ is also prime.

   - $\forall x (P(x) \vee Q(x))$

     False. For example, the number 6 is neither prime nor divisible by 5.

   - $\exists x \exists y (P(x) \vee P(y)) \wedge (\neg P(x) \vee \neg P(y))$

     True. For example, take $x = 3$ and $y = 4$. Then $P(x)$ is true since x is prime and $\neg P(y)$ is true since y is not prime. So $(P(x) \vee P(y))$ is true and $(\neg P(x) \vee \neg P(y))$ is true, so $(P(x) \vee P(y)) \wedge (\neg P(x) \vee \neg P(y))$ is true.