Math 55 Quiz 11 Solutions

April 27, 2016

1. Prove that Q_n is bipartite for all n.

   The vertices of Q_n are labeled with bit strings of length n, with an edge between two vertices if and only if they differ by exactly one bit. Let V_1 be the set of vertices with odd bit-sum (the sum of the digits of the bit string) and V_2 be the set of vertices with even bit-sum. If the bit strings of two vertices differ by exactly one bit then those strings agree everywhere except for on one bit at which one string has a 0 and the other a 1. Thus one of the bit strings has odd bit-sum and one even. This means that all edges in Q_n are between a vertex in V_1 and a vertex in V_2. So Q_n is bipartite.

2. Let e be a cut edge of a graph G and v an endpoint of e. Prove that v is a cut vertex if and only if $\deg(v) > 1$.

   Since v is an endpoint of e, $\deg(v) \geq 1$. Assume that $\deg(v) > 1$ and consider the graph G' obtained by removing v and all its incident edges (including e). Since e is a cut edge, there exist vertices v_1 and v_2 in G such that there is no path between them when e is removed, i.e. every path between them contains e.

   Case 1: neither v_1 nor v_2 is equal to v. Then both v_1 and v_2 are vertices in G' and there is no path between them in G' as e is not an edge of G'. So G' is disconnected and v is a cut vertex.

   Case 2: v_1 or v_2 is equal to v, assume WOLOG $v_1 = v$. Note that since $\deg(v) > 1$, there exists an edge e' that is not e with endpoint at v. Let u be the other endpoint of e'. I claim there is no path between u and v_2 in G'. Indeed, if there was such a path then amending this path with e' would produce a path between $v = v_1$ and v_2 that is contained inside $G - e$, contradicting the fact that e is a cut edge. So G' is disconnected and v is a cut vertex.

   Now we prove the converse. Assume v is a cut vertex. Then if we remove v and all its incident edges from G and call this new graph G', there exist vertices v_1 and v_2 that do not have a path between them in G'. But v_1 and v_2 had a path between them in G (since G is connected), so this path must have contained one of the edges incident to v and thus must have gone through v. Let $v_1 \ldots u_1 e_1 v e_2 u_2 \ldots v_2$ be this path. Then $u_1 \neq u_2$ (if they were equal we could replace $u_1 e_1 v e_2 u_2$ with simply u_1 and have a path between v_1 and v_2 in G'), so $e_1 \neq e_2$. So v is the endpoint of two different edges and hence $\deg(v) > 1$.

3. Can the following lists of integers be the degree sequences of a simple graph? If so, draw such graph. If not, explain why not.

   - $4, 4, 3, 3, 3$

     No, because there are an odd number of vertices of odd degree.

   - $3, 2, 2, 1, 0$

     Yes, a triangle with a line segment sticking out from one vertex and an isolated vertex. (I expect a picture but it is hard to type up a picture)

   - $5, 3, 3, 3, 3, 3$

     Yes, W_5. (Again, I expect a picture but it is hard to type up a picture)