Name: Math 54 Quiz #11
GSI Shawn McDougal

The 4th problem is on the back of the page. Unless otherwise specified, be sure to show work/give justification for all your answers.

For reference, here's the initial-boundary value problem for heat flow modeled in 10.1 and 10.2, as well as the corresponding formal series solutions.

(1) $\frac{\partial u}{\partial t}(x,t) = \beta \frac{\partial^2 u}{\partial t^2}(x,t), \quad 0 < x < L, t > 0$

(2) $u(0,t) = u(L,t) = 0$

(3) $u(x,0) = f(x)$

$(a) \quad u(x,t) = \sum c_n e^{-\beta(n\pi/L)^2 t} \sin(n\pi x/L)$

$(b) \quad f(x) = \sum c_n \sin(n\pi x/L)$

1. Solve the heat flow problem with $\beta = 3, L = \pi$, and $f(x) = \sin x - 5\sin 2x$

Plugging in $L = \pi$ in the formula for *f*(x) we see $c_1 = 1, c_2 = -5$ and all the other coefficients are zero. Plug all this, including $\beta = 3$, into the formula for *u* to get $u(x,t) = e^{-3t}\sin x - 5e^{-12t}\sin 2x$

2. Mark each statement True or False. *No need to justify your answers.*

a. In the heat flow problem, the condition $u(0,t) = u(L,t) = 0$ is an *initial condition*.

False. It's a *boundary condition*

b. To separate variables means to assume the solution is of the form $X(x) + T(t)$

False. The form is $X(x)T(t)$.

c. An initial-boundary problem always has at least one solution.

False. Although a homogeneous PDE (e.g. line (1) in the heat flow model) itself always has the trivial solution, once you include initial and boundary conditions there might be no solution.

d. In the heat flow problem, if the Fourier series for $u(x,t)$ and $f(x)$ converge, then we know our formal solution is a genuine solution.

False. The only guarantee we have that a formal solution is genuine is when the series converges to a function with continuous second partial derivatives. (cf. p. 583, p.586,p.615)

3. a) Find the general solution $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ where $\mathbf{A} = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(Given: Three eigenvectors of **A** are, in no particular order, col(1,0,1), col(1,0,0) and col(1,1,0).)

Call these vectors u_1, u_2, and u_3. Multiplying each vector by the matrix, we see that the corresponding eigenvalues are -1, 1, and 3.

Thus, the general solution is $c_1 e^{-t} u_1 + c_2 e^t u_2 + c_3 e^{3t} u_3$, or, equivalently $\begin{bmatrix} e^{-t} & e^t & e^{3t} \\ 0 & 0 & e^{3t} \\ e^{-t} & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

in the $\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c}$ fundamental matrix form.

b) For the same system, solve the initial value problem: $\mathbf{x}(0) = \text{col}(3,1,1)$

We must solve $\mathbf{x}(0) = \mathbf{X}(0)\mathbf{c} = \text{col}(3,1,1)$ then use these weights for our solution vector. Solving this system $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ gives $c_1 = c_2 = c_3 = 1$, so the solution is

$$\mathbf{x}(t) = \mathbf{X}(t) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-t} + e^t + e^{3t} \\ e^{3t} \\ e^{-t} \end{bmatrix}$$

4. Consider the system $\mathbf{x}'(t) = \mathbf{B}\mathbf{x}(t)$, with $\mathbf{B} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -13 & 4 \end{bmatrix}$.

a) What are the eigenvalues? To get the characteristic polynomial we must find the determinant of

$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ -1 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 13 & \lambda - 4 \end{bmatrix}$. The determinant of a block matrix of the form $\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix}$ is simply $\det(\mathbf{A})\det(\mathbf{B})$. So we get the polynomial $\begin{vmatrix} \lambda & 1 \\ -1 & \lambda \end{vmatrix} \cdot \begin{vmatrix} \lambda & -1 \\ 13 & \lambda - 4 \end{vmatrix} = (\lambda^2 + 1)(\lambda^2 - 4\lambda + 13)$ which has roots $\pm i, 2 \pm 3i$. Note: More generally, the same determinant formula also works for "block triangular matrices" of the form $\begin{bmatrix} \mathbf{A} & * \\ \mathbf{0} & \mathbf{B} \end{bmatrix}$ or $\begin{bmatrix} \mathbf{A} & 0 \\ * & \mathbf{B} \end{bmatrix}$ where * can be any block.

Alternatively, we can expand the determinant along the first row to get

$$\lambda \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 13 & \lambda - 4 \end{vmatrix} - \begin{vmatrix} -1 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 13 & \lambda - 4 \end{vmatrix} = (\lambda^2 + 1) \begin{vmatrix} \lambda & -1 \\ 13 & \lambda - 4 \end{vmatrix} = (\lambda^2 + 1)(\lambda^2 - 4\lambda + 13)$$ for the same result.

b) Choose one of the eigenvalues. Suppose the corresponding eigenvector is $\mathbf{a} + i\mathbf{b}$. In terms of $\mathbf{a}$ and $\mathbf{b}$, write down one of the real-valued solutions to the system. Choose $\lambda = i$ gives us, e.g. $\cos t\mathbf{a} - \sin t\mathbf{b}$ as one real-valued solution. Choosing $\lambda = 2 + 3i$ gives us, e.g. $e^{2t}(\cos 3t\mathbf{a} - \sin 3t\mathbf{b})$