

Math 121A Spring 2007

Homework#11 Fourier transform, Series and Parseval's theorem.

1. Consider the following sets $L^2(T)$ of all functions from $[-\pi, \pi]$ to \mathbb{C} that satisfy $\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx < \infty$ and the set $l^2(\mathbb{Z})$ of all sequences $(a_n)_{n \in \mathbb{Z}}$ of complex numbers such that $\sum_{n \in \mathbb{Z}} |a_n|^2 < \infty$. Show that $L^2(T)$ and $l^2(\mathbb{Z})$ are vector spaces over \mathbb{C} (Clue: Triangle inequality). What is the dimension of these vector spaces?
2. For a function $f \in L^2(T)$ we define its *Fourier transform* \hat{f} to be the sequence $(\hat{f}(n))$ with $\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx} dx$. We define the Fourier series of f to be the series $\sum_{n \in \mathbb{Z}} \hat{f}(n)e^{inx}$. Compute the Fourier series of the following functions
 - (a) $f(x) = x^2$.
 - (b) $f(x) = |x|$.
 - (c) $f(x) = 0$ for $-\pi \leq x \leq 0$ and $f(x) = \sin(x)$ for $0 \leq x \leq \pi$.
3. Consider the inner products $\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)} dx$ on $L^2(T)$ and $[(a_n), (b_n)] = \sum_{n \in \mathbb{Z}} a_n \overline{b_n}$ on $l^2(\mathbb{Z})$. We have the following Theorem

Theorem (Parseval). The Fourier transform $\hat{\cdot}: L^2(T) \rightarrow l^2(\mathbb{Z})$ is an isomorphism of inner product spaces. i.e., Fourier transform is a linear operator, it is 1-1 and onto and for every $f, g \in L^2(T)$ we have $\langle f, g \rangle = [(\hat{f}(n)), (\hat{g}(n))]$.

- Show that in particular for every $f \in L^2(T)$ we have $\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2$ (*)
- Use the identity (*) and the function $f: [-\pi, \pi] \rightarrow \mathbb{C}$, $f(x) = x$ to verify the identity $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$.
- Find what identities follows from (a) above when applied to the functions appearing in 2.(a), 2.(b) and 2.(c) above.
- **Remarks**
 - You are very much encouraged to work with other students. However, submit your work alone.
 - I will be happy to help you with the homeworks. Please visit me if you want to work with me.

Good luck!