Solutions

Sample test #1.

1. Use substitution $u = \sin x$

$$\int \frac{\cos x}{e^{\sin x}+1}dx = \int \frac{du}{1+e^u} = \int \frac{e^u du}{e^u + e^{2u}},$$

now use $z = e^u$

$$\int \frac{e^u du}{e^u + e^{2u}} du = \int \frac{dz}{z+z^2} = \int \frac{1}{z} - \frac{1}{z+1} dz = \ln |\frac{z}{z+1}| + C = \ln \frac{e^{\sin x}}{1+e^{\sin x}} + C$$

2. On the interval $[0, 1]$

$$\frac{1}{x\sqrt{x+1}} \geq \frac{1}{x\sqrt{2}}.$$

Therefore by comparison test

$$\int_0^1 \frac{dx}{x\sqrt{x+1}} \geq \int_0^1 \frac{dx}{x\sqrt{2}}$$

is divergent.

3.

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} n \ln \left(1 + \frac{1}{n^2}\right) = \lim_{x\to 0} \frac{\ln(1+x^2)}{x} = \lim_{x\to 0} \frac{x^2 + \dots}{x} = 0.$$

Hence

$$\lim_{n\to\infty} a_n = 1.$$

4.

$$k = \frac{3}{0.2} = 15.$$

$$W = \int_{0.2}^{0.5} 15x dx = \frac{15}{2} x^2|_{0.2}^{0.5} = \frac{15}{2}(0.25 - 0.04) = 1.575J.$$

5.

$$\lim_{n\to\infty} \frac{n^{1/3}}{n^2+1} \div \frac{1}{n^{5/3}} = 1.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{5/3}}$ converges

$$\sum_{n=1}^{\infty} \frac{n^{1/3}}{n^2+1}$$

converges by limit comparison test. Thus

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^{1/3}}{n^2+1}$$

also converges.

6.

$$\lim_{n\to\infty}\frac{|x|^{n+1}}{3^{n+1}(n+1)^2}\div\frac{|x|^n}{3^n n^2}=\frac{|x|}{3}.$$

The radius of convergence is 3. If $x=3$ or $x=-3$ we get convergent series

$$\sum_{n=0}^{\infty}\frac{1}{n^2},\ \sum_{n=0}^{\infty}\frac{(-1)^n}{n^2}.$$

The interval of convergence is $[-3,3]$.

7.

$$yy'=x\sqrt{1+x^2}\sqrt{1+y^2}.$$

$$\int\frac{ydy}{\sqrt{1+y^2}}=\int x\sqrt{1+x^2}dx$$

$$\sqrt{1+y^2}=\frac{1}{3}\left(1+x^2\right)^{3/2}+C$$

8. Solve the differential equation $y''-2y=e^{2x}$.

$$r^2-2=0,\ r_{1,2}=\pm\sqrt{2},$$

$$y_c=c_1e^{\sqrt{2}x}+c_2e^{-\sqrt{2}x},$$

$$y_p=Ae^{2x},\ 4Ae^{2x}-2Ae^{2x}=e^{2x},\ A=\frac{1}{2},$$

$$y=\frac{1}{2}e^{2x}+c_1e^{\sqrt{2}x}+c_2e^{-\sqrt{2}x}.$$

9.

$$y''-xy=0,\ y(0)=1,\ y'(0)=0,\ y=\sum_{n=0}^{\infty}c_nx^n$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n-\sum_{n=0}^{\infty}c_nx^{n+1}=0$$

$$(n+2)(n+1)c_{n+2}=c_{n-1},\ c_0=1,\ c_1=0,\ c_2=0,c_{3k+1}=c_{3k+2}=0,$$

$$c_{3k}=\frac{1}{3k\cdot(3k-1)\cdot(3k-3)\cdot(3k-4)\cdot\cdots\cdot3\cdot2},$$

$$y=1+\sum_{k=1}^{\infty}\frac{x^{3k}}{3k\cdot(3k-1)\cdot(3k-3)\cdot(3k-4)\cdot\cdots\cdot3\cdot2}.$$

10. $m=3,\ c=11,\ k=\frac{3}{0.3}=10.$

$$3\frac{d^2x}{dt^2}+11\frac{dx}{dt}+10x=0,\ x(0)=0.3,\ x'(0)=0$$

$$3r^2+11r+10=0,\ r_1=-2,\ r_2=-\frac{5}{3}$$

$$y = c_1 e^{-2t} + c_2 e^{-\frac{5}{3}t},\ c_1 + c_2 = 0.3,\ -2c_1 - \frac{5}{3}c_2 = 0,\ c_1 = -\frac{5}{6}c_2,\ c_2 = 1.8,\ c_1 = -1.5$$

$$y = -1.5e^{-2t} + 1.8e^{-\frac{5}{3}t}$$

Sample final #2.

1. Comparison with $\frac{1}{(x+1)^3}$ shows that the integral is convergent.

$$\frac{\mathbf{1}}{(x+1)^2(x+2)} = \frac{Ax+B}{(x+1)^2} + \frac{C}{(x+2)}$$

$$A = -1,\ B = 0,\ C = 1,\ \frac{\mathbf{1}}{(x+1)^2(x+2)} = -\frac{x}{(x+1)^2} + \frac{1}{(x+2)}.$$

$$\int \frac{dx}{(x+1)^2(x+2)} = -\int \frac{xdx}{(x+1)^2} + \int \frac{dx}{(x+2)} = -\frac{1}{x+1} - \ln|x+1| + \ln|x+2| + C$$

$$\int_0^\infty \frac{dx}{(x+1)^2(x+2)} = \ln\frac{x+2}{x+1} - \frac{1}{x+1}|_0^\infty = 1 - \ln 2.$$

2. Use l'Hospital's rule twice

$$\lim_{n\to\infty} \frac{\ln^2 n}{n} = \lim_{x\to\infty} \frac{\ln^2 x}{x} = \lim_{x\to\infty} \frac{2\ln x}{x} = \lim_{x\to\infty} \frac{2}{x} = 0$$

3. $\sum_{n=1}^{\infty} \frac{n^2-n}{n^4+n}$ is convergent by the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

4. Let $f(x) = x^2 e^{x^3}$. Then

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{3n+2}}{n!},\ c_{100} = 0,\ f^{(100)}(0) = 0.$$

5. The equation

$$y' = 0.5 + 0.1 - 0.015y,\ y(0) = 0.$$

$$\frac{y'}{0.6 - 0.015y} = 1,$$

$$\int \frac{dy}{0.6 - 0.015y} = \int dt,\ -\frac{1}{0.015}\ln|0.6 - 0.015y| = t + C,$$

$$0.6 - 0.015y = Ae^{-0.015t},\ y = 40 - 40e^{-0.015t},\ y(60) = 40 - 40e^{-0.9}.$$

6.

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} x^n.$$

$$\lim_{n\to\infty} \frac{((n+1)!)^2}{(2n+2)!}|x|^{n+1} \div \frac{(n!)^2}{(2n)!}|x|^n =$$

$$\lim_{n\to\infty} \frac{(n+1)^2|x|}{(2n+2)(2n+1)} = \frac{|x|}{4}$$

The radius of convergence is 4. When $|x| = 4$, the series

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} 4^n$$

satisfies the condition

$$\frac{a_{n+1}}{a_n} = \frac{4\,(n+1)^2}{(2n+2)\,(2n+1)} = \frac{4n^2+8n+4}{4n^2+6n+2} > 1.$$

Therefore $a_{n+1} > a_n$. The series is divergent by divergence test. The interval of convergence is $(-4, 4)$.

7. Substitute into equation $y'' - xy' - 2y = 0$ a power series.

$$y = \sum_{n=0}^{\infty} c_n x^n,\ \sum_{n=0}^{\infty} c_{n+2}(n+2)\,(n+1)\,x^n - \sum_{n=0}^{\infty} n c_n x^n - 2\sum_{n=0}^{\infty} c_n x^n = 0,$$

$$(n+2)\,(n+1)\,c_{n+2} - (n+2)c_n = 0,\ c_{n+2} = \frac{c_n}{n+1},\ .$$

$$y = A\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n-1)\,(2n-3)\ldots 1} + B\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n)\,(2n-2)\ldots 2}$$

8.

$$y'' - 2y' - 3y = e^x \cos 4x,\ y\,(0) = y'\,(0) = 0.$$

$$r^2 - 2r - 3 = 0,\ (r-3)\,(r+1) = 0,\ r_1 = 3,\ r_2 = -1$$

$$y_c = c_1 e^{3x} + c_2 e^{-x},\ y_p = Ae^x \cos 4x + \mathrm{Be}^x \sin 4x$$

$$y_p' = Ae^x \cos 4x - 4Ae^x \sin 4x + \mathrm{Be}^x \sin 4x + 4\mathrm{Be}^x \cos 4x = (A+4B)\,e^x \cos 4x + (B-4A)\,e^x \sin 4x,$$

$$y_p'' = (A+4B)\,e^x \cos 4x - 4\,(A+4B)\,e^x \sin 4x + (B-4A)\,e^x \sin 4x + 4\,(B-4A)\,\mathrm{Be}^x \cos 4x =$$

$$= (8B - 15A)\,e^x \cos 4x + (-8A - 15B)\,e^x \sin 4x,$$

substitute into equation

$$8B - 15A - 2\,(A+4B) - 3A = 1,\ (-8A - 15B) - 2\,(B - 4A) - 3B = 0$$

$$A = \frac{-1}{20},\ B = 0$$

$$y_p = \frac{-1}{20} e^x \cos 4x,$$

$$y = -\frac{1}{20} e^x \cos 4x + c_1 e^{3x} + c_2 e^{-x}$$

$$y\,(0) = -\frac{1}{20} + c_1 + c_2 = 0,\ y'\,(0) = -\frac{1}{20} + 3c_1 - c_2 = 0.$$

$$c_1 = c_2 = \frac{1}{40}$$

$$y = -\frac{1}{20} e^x \cos 4x + \frac{1}{40} e^{3x} + \frac{1}{40} e^{-x}.$$

9. $y'' - y = \frac{1}{\cosh x}$.

$$y_c = c_1 e^x + c_2 e^{-x},\ y_p = u_1(x)\, e^x + u_2(x)\, e^{-x},$$

$$e^x u_1' + e^{-x} u_2' = 0,\ e^x u_1' - e^{-x} u_2' = \frac{2}{e^x + e^{-x}}$$

$$u_1' = \frac{e^{-x}}{e^x + e^{-x}},\ u_2' = \frac{-e^x}{e^x + e^{-x}}$$

$$u_1 = \int \frac{e^{-x}}{e^x + e^{-x}} dx = -\frac{1}{2} \ln\left(1 + e^{-2x}\right),$$

$$u_2 = \int \frac{-e^x}{e^x + e^{-x}} dx = -\frac{1}{2} \ln\left(1 + e^{2x}\right),$$

$$y = -\frac{1}{2} \ln\left(1 + e^{-2x}\right) e^x - \frac{1}{2} \ln\left(1 + e^{2x}\right) e^{-x} + c_1 e^x + c_2 e^{-x}.$$

10.

$$R_1(0.2) = \frac{\ln''(z)}{2} (0.2)^2$$

for some $1 < z < 1.2$.

$$R_1(0.2) = \frac{-1}{2z^2} (0.2)^2,\ |R_1(0.2)| < \frac{1}{2} (0.2)^2 = 0.02.$$