SOLUTIONS OF SELECTED HOMEWORK PROBLEMS
MATH 252

Problem. Let G be the group of matrices

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

where x, y, z are elements of the finite field $\mathbb{F}_5$. Classify irreducible representations of G over $\mathbb{C}$.

Solution. There are 5 conjugacy classes with one element

$$\begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

for any $y \in \mathbb{F}_5$ and 24 conjugacy classes, each has one representative

$$\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix},$$

for some $x, y \in \mathbb{F}_5$ such that $x \neq 0$ or $y \neq 0$. Let $H = [G, G]$. Then H coincides with the center of G and consists of matrices

$$\begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Ther is 25 one-dimensional representations, obtained from the representation of $G/H = \mathbb{Z}_5 \times \mathbb{Z}_5$. The remaining four representations have dimension 5, and can be obtained by induction from the subgroup K of matrices

$$M_{x,y} = \begin{pmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $u \in \mathbb{F}_5^*$ and $\chi_u(M_{x,y}) = e^{2\pi uyi/5}$. Then $\rho_u = \operatorname{Ind}_K^G \chi_u$ has dimension 5, $\rho_u \ncong \rho_v$ if $v \neq u$, since the action of the center is different. Finally

$$\langle \chi_{\rho_u}, \chi_{\rho_u} \rangle_G = \langle \operatorname{Res}_K \chi_{\rho_u}, \chi_u \rangle_K = \sum_{t \in \mathbb{F}_5} \langle \chi_u^t, \chi_u \rangle_K = 1,$$

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where $\chi_u^t(M_{x,y}) = \chi_u(M_{x,y+xt})$. That proves irreducibility of each ρ_u. Alternatively, one can prove that ρ_u is irreducible by assuming the contrary. Then a subrepresentation must have dimension 1 (divides the order of the group), but this is impossible since $\rho_u(H) \neq 1$.

Problem. Let G be a finite group, r be the number of conjugacy classes in G and s be the number of conjugacy classes in G preserved by the involution $g \to g^{-1}$. Prove that the number of irreducible representations of G over $\mathbb{R}$ is equal to $\frac{r+s}{2}$.

Solution. Let χ be an irreducible character of G over $\mathbb{C}$. If $\chi(g) \in \mathbb{R}$ for all $g \in G$, then there is one irreducible representation of G over $\mathbb{R}$ with character χ (real) or 2χ (quaternionic). If $\chi(g) \notin \mathbb{R}$ at least for one g, then the pair χ and $\bar{\chi}$ produce one irreducible representation of G over $\mathbb{R}$ (complex) with character $\chi + \bar{\chi}$. Hence if m is the number of irreducible representations of G over $\mathbb{R}$ and p is the number of irreducible characters φ such that $\varphi(g) \in \mathbb{R}$ for all g, then $m = p + \frac{r-p}{2} = \frac{r+p}{2}$. Define the linear operator T on the space of class functions by the formula $T\varphi(g) = \varphi(g) + \varphi(g^{-1})$. Then $\operatorname{rk} T = s + \frac{r-s}{2} = \frac{r+s}{2}$. On the other hand, if φ is an irreducible character, then $\bar{\varphi}(g) = \varphi(g^{-1})$, hence $T(\varphi) = \varphi + \bar{\varphi}$. Since irreducible characters form a basis in the space of class function, one obtains $\operatorname{rk} T = m$.

Problem. Let R be the algebra of polynomial differential operators. In other words R is generated by x and $\frac{\partial}{\partial x}$ with relation

$$\frac{\partial}{\partial x} x - x \frac{\partial}{\partial x} = 1.$$

(The algebra R is called the Weyl algebra.) Let $M = \mathbb{C}[x]$ have a structure of R-module in the natural way. Show that $\mathrm{End}_R(M) = \mathbb{C}$, M is an irreducible R-module and the natural map $R \to \mathrm{End}_{\mathbb{C}}(M)$ is not surjective.

Solution. Note that 1 generates M and if $f \in \mathrm{End}_R(M)$ then $f(p) = pf(1)$ for any $p \in M$. But $\frac{\partial}{\partial x}(f(1)) = 0$. Hence $f(1) = c$ for some $c \in \mathbb{C}$. Therefore $\mathrm{End}_R(M) = \mathbb{C}$. On the other hand, every submodule of M contains 1, therefore M is irreducible. Finally, note that every $d \in R$ has a finite-dimensional kernel. Therefore $\mathrm{End}_{\mathbb{C}}(M) \neq R$.

Problem. Let R be the subalgebra of upper triangular matrices in $\mathrm{Mat}_n(\mathbb{C})$. Classify simple and indecomposable projective modules over R and evaluate $\mathrm{Ext}_R^{\cdot}(M, N)$ for all simple M and N.

Solution. Let E_{ij} denote the elementary matrix with 1 in one place. Then primitive idempotents are E_{ii}, $i = 1, \dots, n$. Indecomposable projectives are $P_i = RE_{ii}$. Note that P_i is isomorphic to the maximal submodule of P_{i+1}. Hence simple modules are $S_i = P_i / P_{i-1}$, if we put $P_0 = 0$. Thus, the complex

$$0 \to P_{i-1} \to P_i \to 0$$

is a projective resolution of S_i. Hence $\mathrm{Ext}^k(S_i, S_j) = 0$ if $k > 1$. Now use that $\mathrm{Hom}_R(P_i, S_i) \cong \mathbb{C}$ and $\mathrm{Hom}_R(P_i, S_j) = 0$ if $i \neq j$, because each P_i has a unique simple quotient isomorphic to S_i. Thus, we obtain $\mathrm{Hom}_R(S_i, S_j) = 0$ if $i \neq j$, $\mathrm{Hom}_R(S_i, S_i) = \mathbb{C}$, $\mathrm{Ext}^1(S_i, S_j) = 0$ if $i \neq j+1$, $\mathrm{Ext}^1(S_{i+1}, S_i) = \mathbb{C}$.

Problem. Let Q be a connected quiver and $k(Q)$ be the path algebra of Q. Show that the center of $k(Q)$ is isomorphic either to k, or to $k[x]$, and that the latter happens only in the case when Q is an oriented cycle.

Solution. Let c be an element of the center of $k(Q)$. Without loss of generality we may assume that c is a linear combination of paths of the same length. Assume that there is an element of the center c of non-zero degree (recall that degree is the length of a path). Write $c = \sum c_{ij}$, where $c_{ij} = e_i c e_j$.

First, we claim that $c_{ij} = 0$ if $i \neq j$. Indeed, if $c_{ij} \neq 0$, then $e_i c = c e_i$ implies $e_i c_{ij} \in k(Q) e_i$, which is impossible.

Next, we claim that if $c_{ii} \neq 0$ for one i, then $c_{jj} \neq 0$ for all j. Indeed, assume the opposite, then, since Q is connected, there exists $\gamma = i \to j$ such that either $c_{ii} = 0$, $c_{jj} \neq 0$ or $c_{jj} = 0$, $c_{ii} \neq 0$. In the former case $e_j \gamma c e_i = \gamma c_{ii} = 0$ and $e_j c \gamma e_i = c_{jj}\gamma \neq 0$, which contradicts $\gamma c = c\gamma$. Similarly, in the latter case $e_j \gamma c e_i \neq 0$, $e_j c \gamma e_i = 0$. Contradiction.

Finally, let γ and $\delta \in Q_1$ and $s(\gamma) = s(\delta) = i$. Then $c\gamma = \gamma c = \gamma c_{ii}$ implies $c_{ii} \in k(Q)\gamma$. By the same reason $c_{ii} \in k(Q)\delta$, which implies $\gamma = \delta$. In the same way, if $t(\gamma) = t(\delta)$, then $\gamma = \delta$. Thus, if there is a central c such that $\deg c > 0$, then Q is one oriented cycle.

Assume first, that Q is not an oriented cycle. The any central element c has degree 0, and therefore $c = \sum b_i e_i$. If $\gamma = i \to j$, then $c\gamma = \gamma c$ implies $b_i = b_j$. But Q is connected, hence $b_1 = \cdots = b_n$. That proves that the center of $k(Q)$ is isomorphic to k.

Let $k(Q)$ be one oriented cycle of length n. Since we already proved that a central element c is a combination of cycles, n divides $\deg c$. A central element of degree sn equals $b \sum sn$-cycles, and hence the center is isomorphic to $k[z]$, where z is the sum of all n-cycles.

Problem. Let $\mathrm{Rep}(a, b, c)$ be the space of all representations of the quiver

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with dimension vector (a, b, c). List all orbits in $\mathrm{Rep}(a, b, c)$. Show that there is only one open orbit. Describe the open orbit O_X in terms of decomposition of X into direct sum of indecomposable representations.

Solution. A point $\mathrm{Rep}(a, b, c)$ is a pair of linear operators $P\colon k^a \to k^b$ and $Q\colon k^c \to k^b$. An orbit is determined by three numbers, $p = \mathrm{rk}\,P$, $q = \mathrm{rk}\,Q$ and $r = \dim(\mathrm{Im}\,P \cap \mathrm{Im}\,Q)$, and we have $p \leq \min(a, b)$, $q \leq \min(b, c)$, $r \leq \min(p, q)$. Positive roots corresponding to indecomposable modules are

$\alpha_1 = (1, 0, 0)$, $\alpha_2 = (0, 1, 0)$, $\alpha_3 = (0, 0, 1)$, $\beta_1 = \alpha_1 + \alpha_2$, $\beta_2 = \alpha_2 + \alpha_3$, $\gamma = \alpha_1 + \alpha_2 + \alpha_3$,

and the decomposition of (P, Q) into the sum of indecomposables is

$$(a-p)\,\alpha_1 + (b-p-q+r)\,\alpha_2 + (c-q)\,\alpha_3 + (p-r)\,\beta_1 + (q-r)\,\beta_2 + r\gamma.$$

By X_ν we denote the indecomposable representation of dimension ν. Check that indecomposable projectives are

$$X_{\beta_1} = Ae_1, X_{\alpha_2} = Ae_2,\, X_{\beta_2} = Ae_3,$$

and the projective resolutions of $X_{\alpha_1}, X_{\alpha_3}$ and X_γ are

$$0 \to X_{\alpha_2} \to X_{\beta_1} \to\ 0,\, 0 \to X_{\alpha_2} \to X_{\beta_2} \to\ 0,$$

$$0 \to X_{\alpha_2} \to X_{\beta_1} \oplus X_{\beta_2} \to\ 0.$$

Therefore,

$$\operatorname{Ext}^1(X_{\alpha_1}, X_{\alpha_2}) = \operatorname{Ext}^1(X_{\alpha_3}, X_{\alpha_2}) = \operatorname{Ext}^1(X_{\alpha_1}, X_{\beta_2}) = \operatorname{Ext}^1(X_{\alpha_3}, X_{\beta_1}) = \operatorname{Ext}^1(X_\gamma, X_{\alpha_2}) = k,$$

all other Ext^1 are trivial.

To determine the open orbit we find possible triples of positive roots without mutual extensions.

$$\{\alpha_1, \beta_1, \gamma\}, \{\alpha_3, \beta_2, \gamma\}, \{\alpha_1, \alpha_3, \gamma\}, \{\beta_1, \beta_2, \gamma\}, \{\beta_1, \beta_2, \alpha_2\}.$$

The open orbit in $\operatorname{Rep}(a, b, c)$ is a combination of one of these triples, here $x = (a, b, c)$:

1. If $a \geq b \geq c$, then $x = (a-b)\,\alpha_1 + (b-c)\,\beta_1 + c\gamma$;
2. If $a \leq b \leq c$, then $x = (c-b)\,\alpha_3 + (b-a)\,\beta_1 + a\gamma$;
3. If $a, c \geq b$, then $x = (a-b)\,\alpha_1 + (c-b)\,\alpha_3 + b\gamma$;
4. If $a, c \leq b, a + c \geq b$, then $x = (b-c)\,\beta_1 + (b-a)\,\beta_2 + (a+c-b)\,\gamma$;
5. If $a, c, a + c \leq b$, then $x = a\beta_1 + c\beta_2 + (b-a-c)\,\alpha_2$.