RATIONALITY QUESTIONS AND HECKE ALGEBRA

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1. Splitting field

We assume that k is a field of characteristic 0, $\bar{k}$ is its algebraic closure and G is a finite group.

Let $\chi_1, \dots, \chi_r$ be the characters of all non-isomorphic irreducible representations of G over $\bar{k}$ and $\psi_1, \dots, \psi_s$ be the characters of all non-isomorphic representations of G over k. Recall that if $\rho : G \to GL(V)$ and $\sigma : G \to GL(W)$ are two representations of G over any field of characteristic zero

$$\dim \operatorname{Hom}_G(V, W) = (\chi_\rho, \chi_\sigma).$$

Therefore $(\psi_i, \psi_j) = 0$ if $i \neq j$.

Let

$$R_k(G) = \{\sum_{j=1}^{s} m_j \psi_j | m_j \in \mathbb{Z}\}$$

and

$$R(G) = \{\sum_{i=1}^{r} m_i \chi_i | m_i \in \mathbb{Z}\}.$$

Note that $R_k(G) \subset R(G)$, and both $R_k(G)$ and $R(G)$ have natural structures of commutative rings.

A representation ρ of G over $\bar{k}$ is *rational* over k if $\rho = \sigma_{\bar{k}} := \bar{k} \otimes_k \sigma$ for some representation σ of G over k. This condition is equivalent to $\chi_\rho \in R_k(G)$.

Lemma 1.1. *The following conditions on a field k are equivalent*

(a) every irreducible representation of G over k is absolutely irreducible;

(b) every representation over $\bar{k}$ is rational over k;

(c) $R_k(G) = R(G)$;

(d) the group algebra $k(G)$ is isomorphic to a direct sum of matrix algebras over k.

Exercise. Prove the lemma.

A field k is a *splitting field* for G if it satisfies the conditions of Lemma 1.1.

Theorem 1.2. *For any field k and a group G there exists a finite extension $F \supset k$ such that F is a splitting field for G.*

Date: March 31, 2011.

Proof. Recall that $\bar{k}(G) = \oplus_{i=1}^{r} \operatorname{End}_{\bar{k}}(V_i)$ is a direct sum of matrix algebras. Let E_{ij}^{p} be the standard basis consisting of elementary matrices. Then $E_{ij}^{p} = \sum_{g \in G} c_{ij}^{p}(g) g$ and $g = \sum_{i,j,p} d_{p}^{ij}(g) E_{ij}^{p}$. Then a finite extension F of k containing $c_{ij}^{p}(g)$ and $d_{p}^{ij}(g)$ is a splitting field for G. □

In fact, the stronger result is true.

Theorem 1.3. *Let m be the least common multiple of orders of all elements in G, and F contains a primitive m-th root of 1. Then F is a splitting field for G.*

2. The number of irreducible representations

Lemma 2.1. *Assume that F is such that $\chi_i(g) \in F$ for all $i \leq r$ and $g \in G$. Then $R_F(G)$ is a subgroup of finite index in $R(G)$.*

Proof. Let τ be an irreducible representation of G over F. The operator of projection on isotypic component $p_i = \frac{n_i}{|G|} \sum \chi_i(g^{-1}) \tau_g$ is well defined. By irreducibility of τ it is either zero or non-degenerate. Hence χ_τ is a multiple of χ_i for one i. The statement follows. □

Let us assume now that F is a splitting field of G and F is a finite Galois extension of F. By Γ denote the corresponding Galois group. Note that if ρ is a representation of G over F, then for each $\gamma \in \Gamma$ one can define the representation ρ^γ by twisting the matrix coefficients by action of γ. It is also clear that if ρ is irreducible, then ρ^γ is also irreducible. Thus, Γ acts on $R(G)$ by permuting $\chi_1, \dots, \chi_r$.

Lemma 2.2. *Let $\rho : G \to GL(V)$ be an irreducible representation of G over k. Then there exist an integer n_ρ and $i \leq r$ such that*

$$\chi_\rho = n_\rho \sum_{\gamma \in \Gamma} \chi_i^\gamma.$$

This n_ρ is called the Schur index of ρ.

Proof. Write $\chi_\rho = \sum a_i \chi_i$. Note that χ_ρ is Γ-invariant. Therefore if $\chi_i = \chi_j^\gamma$ for some $\gamma \in \Gamma$, then $a_i = a_j$. Let $a_i \neq 0$ and $\psi = \sum \chi_j$ over the Γ-orbit of χ_i. Then ψ is Γ-invariant and hence $\psi(g) \in k$ for all $g \in G$. Thus, the operator $p = \sum_{g \in G} \psi(g^{-1}) \rho_g$ is an intertwiner. It is not zero as can be easily shown by calculating the trace, therefore it is non-degenerate. Hence it is non-degenerate on V_F. The latter implies that the characters of all irreducible components of ρ_F are not orthogonal to ψ. The statement follows. □

Theorem 2.3. *The number of irreducible representation of G over k equals the number of Γ-orbits in $\{\chi_1, \dots, \chi_r\}$.*

Proof. From the previous lemma we obtain that the number of irreducible representation is not larger than the number of orbits. On the other hand let ρ be an irreducible representation of G over F, and ρ^k be the same representation considered as a representation over k. Then

$$\chi_(\rho^k) = \sum_{\gamma\in\Gamma} \chi.$$

This shows that the number of irreducible representation of G over k is not less than the number of orbits. □

Let us make an additional assumption that F contains all roots of $x^m - 1$. The set of all roots is a cyclic group of order m and and Γ acts on this group by automorphisms. More precisely, if ε is a primitive root then for each $\gamma \in \Gamma$ there exists $r(\gamma)$ relatively prime to m such that $\gamma(\varepsilon) = \varepsilon^{r(\gamma)}$. Introduce the action of Γ on G by setting $g^\gamma = g^{r(\gamma)}$. If ρ is a representation of G, then all eigenvalues of ρ_g are roots of $x^m - 1$. Therefore

$$\chi_\rho^\gamma(g) = \chi_\rho(g^\gamma).$$

Now we consider the action of $\Gamma \times G$ on G, where G acts by conjugation. The orbits under this action are called Γ-classes.

Theorem 2.4. *The number of Γ classes equals the number of irreducible representations of G over k.*

Proof. Let $R(G)^\Gamma$ be the abelian subgroup of $R(G)$ fixed by the Γ-action. Then $\operatorname{rk} R(G)^\Gamma$ coincides with $\dim F \otimes_{\mathbb{Z}} R(G)^\Gamma$. The representation of Γ in $F \otimes_{\mathbb{Z}} R(G)$ is a permutation representation. Hence $\dim F \otimes_{\mathbb{Z}} R(G)^\Gamma$ equals the number of Γ-orbits in $\{\chi_1, \dots, \chi_r\}$. On the other hand, $F \otimes_{\mathbb{Z}} R(G)$ is the subspace of functions on G consisting of function constant on Γ classes. That implies the theorem. □

Example 2.5. Let $k = \mathbb{R}$, $F = \mathbb{C}$, $\Gamma = \mathbb{Z}_2$ with generator σ which is conjugation. It is easy to see that $\sigma(g) = g^{-1}$ for any $g \in G$. Therefore the number of irreducible representation of G over $\mathbb{R}$ equals the number of conjugacy classes stable under σ plus half the number of classes unstable under σ.

3. Hecke algebra and double cosets

Let G be a finite group. The group algebra $k(G)$ is isomorphic to the algebra of function on G with the operation of convolution

$$f_1 * f_2(g) \sum_{s\in G} f_1(gs^{-1}) f_2(s).$$

The isomorphism can be constructed by the map $g \to \delta_g$, where $\delta_g(x) = \delta_{g,x}$. The proof is based on the the simple identity $\delta_g * \delta_h = \delta_{gh}$.

A certain generalization of this construction is so called Hecke algebra. Let H be a subgroup of G, then the induced representation $\operatorname{Ind}_H^G(triv)$ can be identified with the space of $k(X)$ of functions $X = G/H$. The action of G is given by the formula

$$\rho_g f(x) = f(g^{-1}(x)).$$

The Hecke algebra $A(G,H)$ is by definition $\mathrm{End}_G(k(X))$. Note that the ring $\mathrm{End}_k(k(X))$ is isomorphic to the algebra $k(X\times X)$ with operation of convolution

$$F_1 * F_2(x,z) = \sum_{y\in X} F_1(x,y)F_2(y,z).$$

The action of $k(X\times X)$ on $k(X)$ is given by the formula

$$Ff(x) = \sum_{y\in X} F(x,y)f(y).$$

The condition that $F(x,y)$ commutes with ρ_g is equivalent to the following

$$\rho_g(Ff)(x) = \sum_{y\in X} F(g^{-1}x,y)f(y) = F\rho_g f(x) = \sum_{y\in X} F(x,y)f(g^{-1}y)) = \sum_{y\in X} F(x,gy)f(y).$$

This is equivalent to the condition

$$F(x,y) = F(gx,gy).$$

Thus, the Hecke algebra $A(G,H)$ is the subalgebra of $k(X\times X)$ consisting of function constant on G-orbits in $X\times X$ with respect to the diagonal action. These orbits are in bijection with double cosets $H\setminus G/H$. If we associated with each double coset HwH the characteristic function δ_w, then the convolution operation is given by

$$\delta_w * \delta_v = \frac{1}{|H|}\sum m^u_{v,w}\delta_u,$$

where $m^u_{v,w}$ is the cardinality of $Hw^{-1}H\cap HvHu^{-1}$.

Example 3.1. Let $G = GL_n(\mathbb{F}_q)$ and $H = B$ denote the subgroup of upper triangular matrices. The double cosets are BwB, where w is some permutation matrix. Thus, $\dim A(G,B) = n!$. Let $s_1,\dots,s_{n-1}$ be the characteristic function of double cosets corresponding to the transpositions $(1,2),\dots,(n-1,n)$. By direct computation one can check that the following relations hold

$$s_i * s_i = (q-1)s_i + q,$$
$$s_i * s_j = s_j * s_i, \text{ if } |i-j| \neq 1,$$
$$s_i * s_{i+1} * s_i = s_{i+1} * s_i * s_{i+1}.$$

Indeed, the second relation is straightforward. The first follows from analogous relation for $n=2$, and the last from analogous relation for $n=3$. If now one considers q as a formal parameter one gets a deformation of $k(S_n)$.