REPRESENTATION THEORY
WEEK 9

1. Jordan-Hölder theorem and indecomposable modules

Let M be a module satisfying ascending and descending chain conditions (ACC and DCC). In other words every increasing sequence submodules $M_1 \subset M_2 \subset \dots$ and any decreasing sequence $M_1 \supset M_2 \supset \dots$ are finite. Then it is easy to see that there exists a finite sequence

$$M = M_0 \supset M_1 \supset \dots \supset M_k = 0$$

such that M_i/M_{i+1} is a simple module. Such a sequence is called a Jordan-Hölder series. We say that two Jordan Hölder series

$$M = M_0 \supset M_1 \supset \dots \supset M_k = 0,\ M = N_0 \supset N_1 \supset \dots \supset N_l = 0$$

are equivalent if $k = l$ and for some permutation s $M_i/M_{i+1} \cong N_{s(i)}/N_{s(i)+1}$.

Theorem 1.1. *Any two Jordan-Hölder series are equivalent.*

Proof. We will prove that if the statement is true for any submodule of M then it is true for M. (If M is simple, the statement is trivial.) If $M_1 = N_1$, then the statement is obvious. Otherwise, $M_1 + N_1 = M$, hence $M/M_1 \cong N_1/(M_1 \cap N_1)$ and $M/N_1 \cong M_1/(M_1 \cap N_1)$. Consider the series

$$M = M_0 \supset M_1 \supset M_1 \cap N_1 \supset K_1 \supset \dots \supset K_s = 0,\ M = N_0 \supset N_1 \supset N_1 \cap M_1 \supset K_1 \supset \dots \supset K_s = 0.$$

They are obviously equivalent, and by induction assumption the first series is equivalent to $M = M_0 \supset M_1 \supset \dots \supset M_k = 0$, and the second one is equivalent to $M = N_0 \supset N_1 \supset \dots \supset N_l = \{0\}$. Hence they are equivalent. $\square$

Thus, we can define a length $l(M)$ of a module M satisfying ACC and DCC, and if M is a proper submodule of N, then $l(M) < l(N)$.

A module M is *indecomposable* if $M = M_1 \oplus M_2$ implies $M_1 = 0$ or $M_2 = 0$.

Lemma 1.2. *Let M and N be indecomposable, $\alpha \in \mathrm{Hom}_R(M, N)$, $\beta \in \mathrm{Hom}_R(N, M)$ be such that $\beta \circ \alpha$ is an isomorphism. Then α and β are isomorphisms.*

Proof. We claim that $N = \mathrm{Im}\,\alpha \oplus \mathrm{Ker}\,\beta$. Indeed, $\mathrm{Im}\,\alpha \cap \mathrm{Ker}\,\beta = 0$ and for any $x \in N$ one can write $x = y + z$, where $y = \alpha \circ (\beta \circ \alpha)^{-1} \circ \beta(x)$, $z = x - y$. Then since N is indecomposable, $\mathrm{Im}\,\alpha = N$, $\mathrm{Ker}\,\beta = 0$ and $N \cong M$. $\square$

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Lemma 1.3. *Let M be indecomposable module of finite length and $\varphi \in \mathrm{End}_R(M)$, then either φ is an isomorphism or φ is nilpotent.*

Proof. There is $n > 0$ such that $\mathrm{Ker}\,\varphi^n = \mathrm{Ker}\,\varphi^{n+1}$, $\mathrm{Im}\,\varphi^n = \mathrm{Im}\,\varphi^{n+1}$. In this case $\mathrm{Ker}\,\varphi^n \cap \mathrm{Im}\,\varphi^n = 0$ and hence $M \cong \mathrm{Ker}\,\varphi^n \oplus \mathrm{Im}\,\varphi^n$. Either $\mathrm{Ker}\,\varphi^n = 0$, $\mathrm{Im}\,\varphi^n = M$ or $\mathrm{Ker}\,\varphi^n = M$. Hence the lemma. □

Lemma 1.4. *Let M be as in Lemma 1.3 and $\varphi, \varphi_1, \varphi_2 \in \mathrm{End}_R(M)$, $\varphi = \varphi_1 + \varphi_2$. If φ is an isomorphism then at least one of φ_1, φ_2 is also an isomorphism.*

Proof. Without loss of generality we may assume that $\varphi = id$. But in this case φ_1 and φ_2 commute. If both φ_1 and φ_2 are nilpotent, then $\varphi_1 + \varphi_2$ is nilpotent, but this is impossible as $\varphi_1 + \varphi_2 = \mathrm{id}$. □

Corollary 1.5. *Let M be as in Lemma 1.3. Let $\varphi = \varphi_1 + \cdots + \varphi_k \in \mathrm{End}_R(M)$. If φ is an isomorphism then φ_i is an isomorphism at least for one i.*

It is obvious that if M satisfies ACC and DCC then M has a decomposition

$$M = M_1 \oplus \cdots \oplus M_k,$$

where all M_i are indecomposable.

Theorem 1.6. *(Krull-Schmidt) Let M be a module of finite length and*

$$M = M_1 \oplus \cdots \oplus M_k = N_1 \oplus \cdots \oplus N_l$$

for some indecomposable M_i and N_j. Then $k = l$ and there exists a permutation s such that $M_i \cong N_{s(j)}$.

Proof. Let $p_i : M_1 \to N_i$ be the restriction to M_1 of the natural projection $M \to N_i$, and $q_j : N_j \to M_1$ be the restriction to N_j of the natural projection $M \to M_1$. Then obviously $q_1 p_1 + \cdots + q_l p_l = \mathrm{id}$, and by Corollary 1.5 there exists i such that $q_i p_i$ is an isomorphism. Lemma 1.2 implies that $M_1 \cong N_i$. Now one can easily finish the proof by induction on k. □

2. Some facts from homological algebra

The complex is the graded abelian group $C_\cdot = \oplus_{i \geq 0} C_i$. We will assume later that all C_i are R-modules for some ring R. A differential is an R-morphism of degree -1 such that $d^2 = 0$. Usually we realize $C_\cdot$ by the picture

$$\xrightarrow{d} \cdots \to C_1 \xrightarrow{d} C_0 \to \ 0.$$

We also consider d of degree 1, in this case the superindex $C^\cdot$ and

$$0 \to C^0 \xrightarrow{d} C^1 \xrightarrow{d} \cdots$$

All the proofs are similar for these two cases.

Homology group is $H_i(C) = (\mathrm{Ker}\, d \cap C_i) / dC_{i+1}$.

Given two complexes $(C_\cdot, d)$ and $(C'_\cdot, d')$. A morphism $f : C_\cdot \to C'_\cdot$ preserving grading and satisfying $f \circ d = d' \circ f$ is called a *morphism of complexes*. A morphism of complexes induces the morphism $f_* \colon H_\cdot (C) \to H_\cdot (C')$.

Theorem 2.1. *(Long exact sequence). Let*

$$0 \to C_\cdot \xrightarrow{g} C'_\cdot \xrightarrow{f} C''_\cdot \to 0$$

be a short exact sequence, then the long exacts sequence

$$\xrightarrow{\delta} H_i (C) \xrightarrow{g_*} H_i (C') \xrightarrow{f_*} H_i (C'') \xrightarrow{\delta} H_{i-1} (C) \xrightarrow{g_*} \dots$$

where $\delta = g^{-1} \circ d' \circ f^{-1}$, is exact.

Let $f, g : C_\cdot \to C'_\cdot$ be two morphisms of complexes. We say that f and g are *homotopically equivalent* if there exists $h : C_\cdot \to C'_\cdot (+1)$ (the morphism of degree 1) such that $f - g = h \circ d + d' \circ h$.

Lemma 2.2. *If f and g are homotopically equivalent then $f_* = g_*$.*

Proof. Let $\phi = f - g$, $x \in C_i$ and $dx = 0$. Then

$$\phi (x) = h (dx) + d' (hx) = d' (hx) \in \operatorname{Im} d'.$$

Hence $f_* - g_* = 0$. □

We say that complexes $C_\cdot$ and $C'_\cdot$ are homotopically equivalent if there exist $f : C_\cdot \to C'_\cdot$ and $g : C'_\cdot \to C_\cdot$ such that $f \circ g$ is homotopically equivalent to $\mathrm{id}_{C'}$ and $g \circ f$ is homotopically equivalent to id_C. Lemma 2.2 implies that homotopically equivalent complexes have isomorphic homology. The following Lemma is straightforward.

Lemma 2.3. *If $C_\cdot$ and $C'_\cdot$ are homotopically equivalent then the complexes $\operatorname{Hom}_R (C_\cdot, B)$ and $\operatorname{Hom}_R (C'_\cdot)$ are also homotopically equivalent.*

Note that the differential in $\operatorname{Hom}_R (C_\cdot, B)$ has degree 1.

3. Projective modules

An R-module P is *projective* if for any surjective morphism $\phi : M \to N$ and any $\psi : P \to N$ there exists $f : P \to M$ such that $\psi = \phi \circ f$.

Example. A free module is projective. Indeed, let $\{e_i\}_{i \in I}$be the set of free generators of a free module F, i.e. $F = \oplus_{i \in I} Re_i$. Define $f : F \to M$ by $f (e_i) = \phi^{-1} (\psi (e_i))$.

Lemma 3.1. *The following conditions on a module P are equivalent*

1. *P is projective;*
2. *There exists a free module F such that $F \cong P \oplus P'$;*
3. *Any exact sequence $0 \to N \to M \to P \to 0$ splits.*

Proof. (1) $\Rightarrow$ (3) Consider the exact sequence

$$0 \to N \xrightarrow{\varphi} M \xrightarrow{\psi} P \to 0,$$

then since ψ is surjective, there exists $f : P \to M$ such that $\psi \circ f = \mathrm{id}_P$.

(3) $\Rightarrow$ (2) Every module is a quotient of a free module. Therefore we just have to apply (3) to the exact sequence

$$0 \to N \to F \to P \to 0$$

for a free module F.

(2) $\Rightarrow$ (1) Let $\phi : M \to N$ be surjective and $\psi : P \to N$. Choose a free module F so that $F = P \oplus P'$. Then extend ψ to $F \to N$ in the obvious way and let $f : F \to M$ be such that $\phi \circ f = \psi$. Then the last identity is true for the restriction of f to P. □

A *projective resolution* of M is a complex $P_.$ of projective modules such that $H_i(P_.) = 0$ for $i > 0$ and $H_0(P_.) \cong M$. A projective resolution always exists since one can easily construct a resolution by free modules. Below we prove the "uniqueness" statement.

Lemma 3.2. *Let $P_.$ and $P'_.$ be two projective resolutions of the same module M. Then there exists a morphism $f : P_. \to P'_.$ of complexes such that $f_* : H_0(P_.) \to H_0(P'_.)$ induces the identity id_M. Any two such morphisms f and g are homotopically equivalent.*

Proof. Construct f inductively. Let $p : P_0 \to M$ and $p' : P'_0 \to M$ be the natural projections, define $f : P_0 \to P'_0$ so that $p' \circ f = p$. Then

$$f(\operatorname{Ker} p) \subset \operatorname{Ker} p', \ \operatorname{Ker} p = d(P_1), \ \operatorname{Ker} p' = d'(P'_1),$$

hence $f \circ d(P_1) \subset d'(P'_1)$, and one can construct $f : P_1 \to P'_1$ such that $f \circ d = d' \circ f$. Proceed in the same manner to construct $f : P_i \to P_i$.

To check the second statement, let $\varphi = f - g$. Then $p' \circ \varphi = 0$. Hence

$$\varphi(P_0) \subset \operatorname{Ker} p' = d'(P'_1).$$

Therefore one can find $h : P_0 \to P'_1$ such that $d' \circ h = \varphi$. Furthermore,

$$d' \circ h \circ d = \varphi \circ d = d' \circ \varphi,$$

hence

$$(\varphi - h \circ d)(P_1) \subset P'_1 \cap \operatorname{Ker} d' = d'(P'_2).$$

Thus one can construct $h : P_1 \to P'_2$ such that $d' \circ h = \varphi - h \circ d$. Then proceed inductively to define $h : P_i \to P'_{i+1}$. □

Corollary 3.3. *Every two projective resolutions of M are homotopically equivalent.*

Let M and N be two modules and $P_{\cdot}$ be a projective resolution of M. Consider the complex

$$0 \to \mathrm{Hom}_R(P_0, N) \to \mathrm{Hom}_R(P_1, N) \to \dots,$$

where the differential is defined naturally. The cohomology of this complex is denoted by $\mathrm{Ext}_R^{\cdot}(M, N)$. Lemma 2.3 implies that $\mathrm{Ext}_R^{\cdot}(M, N)$ does not depend on a choice of projective resolution for M. Check that $\mathrm{Ext}_R^0(M, N) = \mathrm{Hom}_R(M, N)$.

Example 1. Let $R = \mathbb{C}[x]$ be the polynomial ring. Any simple R-module is one-dimensional and isomorphic to $\mathbb{C}[x]/(x-\lambda)$. Denote such module by $\mathbb{C}_\lambda$. A projective resolution of $\mathbb{C}_\lambda$ is

$$0 \to \mathbb{C}[x] \xrightarrow{d} \mathbb{C}[x] \to 0,$$

where $d(1) = x - \lambda$. Let us calculate $\mathrm{Ext}^{\cdot}(\mathbb{C}_\lambda, \mathbb{C}_\mu)$. Note that $\mathrm{Hom}_{\mathbb{C}[x]}(\mathbb{C}_\mu) = \mathbb{C}$, hence we have the complex

$$0 \to \mathbb{C} \xrightarrow{d^*} \mathbb{C} \to 0$$

where $d^* = \lambda - \mu$. Hence $\mathrm{Ext}^{\cdot}(\mathbb{C}_\lambda, \mathbb{C}_\mu) = 0$ if $\lambda \neq \mu$ and $\mathrm{Ext}^0(\mathbb{C}_\lambda, \mathbb{C}_\lambda) = \mathrm{Ext}^1(\mathbb{C}_\lambda, \mathbb{C}_\lambda) = \mathbb{C}$.

Example 2. Let $R = \mathbb{C}[x]/(x^2)$. Then R has one up to isomorphism simple module, denote it by $\mathbb{C}_0$. A projective resolution for $\mathbb{C}_0$ is

$$\dots \xrightarrow{d} R \xrightarrow{d} R \to 0,$$

where $d(1) = x$ and $\mathrm{Ext}^i(\mathbb{C}_0, \mathbb{C}_0) = \mathbb{C}$ for all $i \geq 0$.

4. Representations of artinian rings

An *artinian* ring is a unital ring satisfying the descending chain condition for left ideals. We will see that an artinian ring is a finite length module over itself. Therefore R is automatically noetherian. A typical example of an artinian ring is a finite-dimensional algebra over a field.

Theorem 4.1. *Let R be an artinian ring, $I \subset R$ be a left ideal. If I is not nilpotent, then I contains an idempotent.*

Proof. Let J be a minimal left ideal, such that $J \subset I$ and J is not nilpotent. Then $J^2 = J$. Let L be a minimal left ideal such that $L \subset J$ and $JL \neq 0$. Then there is $x \in L$ such that $Jx \neq 0$. But then $Jx = L$ by minimality of L. Thus, for some $r \in R$, $rx = x$, hence $r^2x = rx$ and $(r^2 - r)x = 0$. Let $N = \{y \in J \mid yx = 0\}$. Then N is a proper left ideal in J and therefore N is nilpotent. Thus, we obtain

$$r^2 \equiv r \mod N.$$

Let $n = r^2 - r$, then

$$(r + n - 2rn)^2 \equiv r^2 + 2rn - 4r^2n \mod N^2,$$
$$r^2 + 2rn - 4r^2n \equiv r + n - 2rn \mod N^2.$$

Hence $r_1 = r + n - 2rn$ is an idempotent modulo N^2. Repeating this process several times we obtain an idempotent. $\square$

Corollary 4.2. *If an artinian ring does not have nilpotent ideals, then it is semisimple.*

Proof. The sum S of all minimal left ideals is semisimple. By DCC S is a finite direct sum of minimal left ideals. Then S contains an idempotent e, which is the sum of idempotents in each direct summand. Then $R = S \oplus R(1-e)$, however that implies $R = S$. $\square$

Important notion for a ring is the *radical*. For an R-module M let

$$\operatorname{Ann} M = \{x \in R \mid xM = 0\}.$$

Then the radical $\operatorname{rad} R$ is the intersection of $\operatorname{Ann} M$ for all simple R-modules M.

Theorem 4.3. *If R is artinian then $\operatorname{rad} R$ is a maximal nilpotent ideal.*

Proof. First, let us show that $\operatorname{rad} R$ is nilpotent. Assume the contrary. Then $\operatorname{rad} R$ contains an idempotent e. But then e does not act trivially on a simple quotient of Re. Contradiction.

Now let us show that any nilpotent ideal N lies in $\operatorname{rad} R$. Let M be a simple module, then $NM \neq M$ as N is nilpotent. But NM is a submodule of M. Therefore $NM = 0$. Hence $N \subset \operatorname{Ann} M$ for any simple M. $\square$

Corollary 4.4. *An artinian ring R is semisimple iff $\operatorname{rad} R = 0$.*

Corollary 4.5. *If R is artinian, then $R/\operatorname{rad} R$ is semisimple.*

Corollary 4.6. *If R is artinian and M is an R-module, then for the filtration*

$$M \supset (\operatorname{rad} R)\, M \supset (\operatorname{rad} R)^2\, M \supset \cdots \supset (\operatorname{rad} R)^k\, M = 0$$

all quotients are semisimple. In particular, M always has a simple quotient.

Theorem 4.7. *If R is artinian, then it has finite length as a left module over itself.*

Proof. Consider the filtration $R = R_0 \supset R_1 \supset \cdots \supset R_s = 0$ where $R_i = (\operatorname{rad} R)^i$. Then each quotient R_i/R_{i+1} is semisimple of finite length. The statement follows. $\square$

Let R be an artinian ring. By Krull-Schmidt theorem R (as a left module over itself) has a decomposition into direct sum of indecomposable submodules $R = L_1 \oplus \cdots \oplus L_n$. Since $\operatorname{End}_R(R) = R^{\mathrm{op}}$, the projector on each component L_i is given by multiplication on the right by some idempotent e_i. Thus, $R = Re_1 \oplus \cdots \oplus Re_n$, where e_i are idempotents and $e_i e_j = 0$ if $i \neq j$. This decomposition is unique up to multiplication on some unit on the right. Since Re_i is indecomposable, e_i can not be written as a sum of two orthogonal idempotents, such idempotents are called *primitive*. Each module Re_i is projective.

Lemma 4.8. *Let R be artinian, $N = \operatorname{rad} R$ and e be a primitive idempotent. Then Ne is a unique maximal submodule of Re.*

Proof. Since Re is indecomposable, every proper left ideal is nilpotent, (otherwise it has an idempotent and therefore splits as a direct summand in Re). But then this ideal is in $N \cap Re = Ne$. □

A projective module P is a *projective cover* of M if there exists a surjection $P \to M$.

Theorem 4.9. *Let R be artinian. Every simple R-modules S has a unique (up to an isomorphism) indecomposable projective cover isomorphic to Re for some primitive idempotent $e \in R$. Every indecomposable projective module has a unique (up to an isomorphism)simple quotient.*

Proof. Every simple S is a quotient of R, and therefore it is a quotient of some indecomposable projective $P = Re$. Let $\phi : P \to S$ be the natural projection. For any indecomposable projective cover P_1 of S with surjective morphism $\phi_1 : P_1 \to S$ there exist $f : P \to P_1$ and $g : P_1 \to P$ such that $\phi = \phi_1 \circ f$ and $\phi_1 = \phi \circ g$. Therefore $\phi = \phi \circ g \circ f$. Since $\operatorname{Ker} \phi$ is the unique maximal submodule, $g \circ f(P_1) = P$. In particular g is surjective. Indecomposability of P_1 implies $P \cong P_1$. Thus, every simple module has a unique indecomposable projective cover.

On the other hand, let P be an indecomposable projective module. Corollary 4.6 implies that P has a simple quotient S. Hence P is isomorphic to the indecomposable projective cover of S. □

Corollary 4.10. *Every indecomposable projective module over an artinian ring R is isomorphic to Re for some primitive idempotent $e \in R$. There is a bijection between the ismorphism classes of simple R-modules and isomorphism classes of projective indecomposable R-modules.*

Example. Let $R = \mathbb{F}_3(S_3)$. Let r be a 3-cycle and s be a transposition. Then $r-1$, r^2-1, $sr-s$ and sr^2-s span a maximal nilpotent ideal. Hence R has two (up to an isomorphism) simple modules L_1 and L_2, where L_1 is a trivial representation of S_3 and L_2 is a sign representation. Choose primitive idempotents $e_1 = -s-1$ and $e_2 = s-1$, then $1 = e_1 + e_2$. Hence R has two indecomposable projective modules $P_1 = Re_1$ and $P_2 = Re_2$. Note that

$$Re_1 \cong \operatorname{Ind}_{S_2}^{S_3}(\text{triv}), \; Re_2 \cong \operatorname{Ind}_{S_2}^{S_3}(\text{sgn}).$$

Thus, P_1 is just 3-dimensional permutation representation of S_3, and P_2 is obtained from P_1 by tensoring with sgn. It is easy to see that P_1 has a trivial submodule as well as a trivial quotient, and sgn is isomorphic to the quotient of the maximal submodule of P_1 by the trivial submodule. One can easily get a similar description for P_2 Thus, one has the the following exact sequences

$$0 \to L_2 \to P_2 \to P_1 \to L_1 \to 0, \qquad 0 \to L_1 \to P_1 \to P_2 \to L_2 \to 0,$$

therefore
$$\cdots \to P_1 \to P_2 \to P_2 \to P_1 \to P_1 \to P_2 \to P_2 \to P_1 \to 0$$
is a projective resolution for L_1, and
$$\cdots \to P_2 \to P_1 \to P_1 \to P_2 \to P_2 \to P_1 \to P_1 \to P_2 \to 0$$
is a projective resolution for L_2. Now one can calculate Ext between simple modules
$$\operatorname{Ext}^k(L_i, L_i) = 0 \text{ if } k \equiv 1, 2 \mod 4, \qquad \operatorname{Ext}^k(L_i, L_i) = \mathbb{F}_3 \text{ if } k \equiv 0, 3 \mod 4,$$
and if $i \neq j$, then
$$\operatorname{Ext}^k(L_i, L_j) = 0 \text{ if } k \equiv 0, 3 \mod 4, \qquad \operatorname{Ext}^k(L_i, L_j) = \mathbb{F}_3 \text{ if } k \equiv 1, 2 \mod 4.$$