REPRESENTATION THEORY
WEEK 7

1. Characters of GL_k and S_n

A character of an irreducible representation of GL_k is a polynomial function constant on every conjugacy class. Since the set of diagonalizable matrices is dense in GL_k, a character is defined by its values on the subgroup of diagonal matrices in GL_k. Thus, one can consider a character as a polynomial function of $x_1, \dots, x_k$. Moreover, a character is a symmetric polynomial of $x_1, \dots, x_k$ as the matrices $\operatorname{diag}(x_1, \dots, x_k)$ and $\operatorname{diag}\left(x_{s(1)}, \dots, x_{s(k)}\right)$ are conjugate for any $s \in S_k$.

For example, the character of the standard representation in E is equal to $x_1 + \dots + x_k$ and the character of $E^{\otimes n}$ is equal to $(x_1 + \dots + x_k)^n$.

Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 0$. Let D_λ denote the determinant of the $k \times k$-matrix whose i, j entry equals $x_i^{\lambda_j}$. It is clear that D_λ is a skew-symmetric polynomial of $x_1, \dots, x_k$. If $\rho = (k-1, \dots, 1, 0)$ then $D_\rho = \prod_{i \leq j} (x_i - x_j)$ is the well known Vandermonde determinant. Let

$$S_\lambda = \frac{D_{\lambda+\rho}}{D_\rho}.$$

It is easy to see that S_λ is a symmetric polynomial of $x_1, \dots, x_k$. It is called a *Schur polynomial*. The leading monomial of S_λ is the $x^{\lambda_1} \dots x_k^{\lambda_k}$ (if one orders monomials lexicographically) and therefore it is not hard to show that S_λ form a basis in the ring of symmetric polynomials of $x_1, \dots, x_k$.

Theorem 1.1. *The character of W_λ equals to S_λ.*

I do not include a proof of this Theorem since it uses beautiful but hard combinatoric. The proof is much easier in general framework of Lie groups and is included in 261A course.

Exercise. Check that

$$\dim W_\lambda = \prod_{i<j} \frac{\left(\bar{\lambda}_i - \bar{\lambda}_j\right)}{(\rho_i - \rho_j)} = \frac{\prod_{i<j} \left(\bar{\lambda}_i - \bar{\lambda}_j\right)}{(k-1)!\,(k-2)!\dots 1!},$$

if $\bar{\lambda} = \lambda + \rho$.

Now we use Schur-Weyl duality to establish the relation between characters of S_n and GL_k. Recall that the conjugacy classes in S_n are given by partitions of n. Let $C(\mu)$ be the class associated with the partition μ in the natural way. Let ρ denote

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the representation of $S_n \times \mathrm{GL}_k$ in $E^{\otimes n}$. Let r be the number of rows in μ. Then one can see that

$$\mathrm{tr}\left(\rho_{s\times g}\right) = \left(x_1^{\mu_1} + \dots + x_k^{\mu_1}\right)\dots\left(x_1^{\mu_r} + \dots + x_k^{\mu_r}\right), \tag{1.1}$$

for any $s \in C(\mu)$ and a diagonal $g \in \mathrm{GL}_k$. Denote by P_μ the polynomial in the right hand side of the identity. Let χ_λ be the character of V_λ. Since

$$\mathrm{tr}\left(\rho_{s\times g}\right) = \sum_{\lambda\in\Gamma_{n,k}} \chi_\lambda(s)\, S_\lambda(g),$$

one obtains the following remarkable relation

$$P_\mu = \sum_{\lambda\in\Gamma_{n,k}} \chi_\lambda(s)\, S_\lambda. \tag{1.2}$$

2. Representations of compact groups

Let G be a group and a topological space. We say that G is a *topological group* if the multiplication map $G \times G \to G$ and the inverse $G \to G$ are continuous maps. Naturally, G is compact if it is compact topological space.

Examples. The circle

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$

A torus $T^n = S^1 \times \dots \times S^1$.

Unitary group

$$U_n = \left\{X \in \mathrm{GL}_n \mid \bar{X}^t X = 1_n\right\}.$$

Special unitary group

$$SU_n = \{X \in U_n \mid \det X = 1\}.$$

Orthogonal group

$$O_n = \left\{x \in \mathrm{GL}_n(\mathbb{R}) \mid X^t X = 1_n\right\}.$$

Special orthogonal group

$$SO_n = \{X \in O_n \mid \det X = 1\}.$$

Theorem 2.1. *Let G be a compact group. There exists a unique measure on G such that*

$$\int_G f(ts)\,dt = \int_G f(t)\,dt,$$

for any integrable function f on G and any $s \in G$, and $\int_G dt = 1$.

In the same way there exists a measure $d't$ such that

$$\int_G f(st)\,dt = \int_G f(t)\,d't, \int_G d't = 1.$$

Moreover, for a compact group $dt = d't$.

The measure dt $(d't)$ is called right-invariant (left-invariant) measure, or Haar measure.

We do not give the proof of this theorem in general. However, all examples we consider are smooth submanifolds in GL_k. Thus, to define the invariant measure we just need to define a volume in the tangent space at identity T_1G and then use right (left) multiplication to define it on the whole group. More precisely, let $\gamma \in \Lambda^{\mathrm{top}} T_1^* G$. Then

$$\gamma_s = m_s^* (\gamma),$$

where $m_s : G \to G$ is the right (left) multiplication on s and m_s^* is the induced map $\Lambda^{\mathrm{top}} T_1^* G \to \Lambda^{\mathrm{top}} T_s^* G$. After this normalize γ to satisfy $\int_G \gamma = 1$.

Consider a vector space over $\mathbb{C}$ equipped with topology such that addition and multiplication by a scalar are continuous. We always assume that a topological vector space satisfies the following conditions

(1) for any $v \in V$ there exist a neighborhood of 0 which does not contain v;
(2) there is a base of convex neighborhoods of zero.

Topological vector spaces satisfying above conditions are called *locally convex.* We do not go into the theory of such spaces. All we need is the fact that there is a non-zero continuous linear functional on a locally convex space.

A representation $\rho : G \to \mathrm{GL}(V)$ is continuous if the map $G \times V \to V$ given by $(s, v) \mapsto \rho_s v$ is continuous.

Regular representation. Let G be a compact group and $L^2(G)$ be the space of all complex valued functions on G such that

$$\int |f(t)|^2 dt$$

exists. Then $L^2(G)$ is a Hilbert space with respect to Hermitian form

$$\langle f, g \rangle = \int_G \bar{f}(t)\, g(t)\, dt.$$

Moreover, a representation R of G in $L^2(G)$ given by

$$R_s f(t) = f(ts)$$

is continuous and the Hermitian form is G-invariant.

A representation $\rho : G \to \mathrm{GL}(V)$ is called *topologically irreducible* if any invariant closed subspace of V is either V or 0.

Lemma 2.2. *Every irreducible representation of G is isomorphic to a subrepresentation in $L^2(G)$.*

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be irreducible. Pick a non-zero linear functional φ on V and define the map $\Phi : V \to L^2(G)$ which sends v to the matrix coefficient $f_{v,\varphi}(s) = \langle \varphi, \rho_s v \rangle$. It is clear that a matrix coefficient is a continuous function on G, therefore $f_{v,\varphi} \in L^2(G)$. Furthermore Φ is a continuous intertwiner and $\mathrm{Ker}\, \Phi = 0$. □

Recall that a Hilbert space is a space over $\mathbb{C}$ equipped with positive definite Hermitian form $\langle , \rangle$ complete in topology defined by the norm

$$\|v\| = \langle v, v\rangle^{1/2}.$$

We need the fact that a Hilbert space has an orthonormal topological basis. A continuous representation $\rho : G \to \mathrm{GL}(V)$ is called *unitary* if V is a Hilbert space and

$$\langle v, v\rangle = \langle \rho_g v, \rho_g v\rangle$$

for any $v \in V$ and $g \in G$. The regular representation of G in $L^2(G)$ is unitary. In fact, Lemma 2.2 implies

Corollary 2.3. *Every topologically irreducible representation of a compact group G is a subrepresentation in $L^2(G)$.*

Lemma 2.4. *Every irreducible unitary representation of a compact group G is finite-dimensional.*

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible unitary representation. Choose $v \in V$, $\|v\| = 1$. Define an operator $T : V \to V$ by the formula

$$Tx = \langle v, x\rangle v.$$

One can check easily that T is self-adjoint, i.e.

$$\langle x, Ty\rangle = \langle Tx, y\rangle.$$

Let

$$\bar{T}x = \int_G \rho_g T\left(\rho_g^{-1}x\right) dg.$$

Then $\bar{T} : V \to V$ is an intertwiner and a self-adjoint operator. Furthermore, $\bar{T}$ is compact, i.e. if

$$S = \{x \in V \mid \|x\| = 1\},$$

then $\bar{T}(S)$ is a compact set in V. Every self-adjoint compact operator has an eigenvector. To construct an eigen vector find $x \in S$ such that $|\left(\bar{T}x, x\right)|$ is maximal. Then $\bar{T}x = \lambda x$. Since $\mathrm{Ker}\left(\bar{T} - \lambda\,\mathrm{Id}\right)$ is an invariant subspace in V, $\mathrm{Ker}\left(\bar{T} - \lambda\,\mathrm{Id}\right) = 0$. Hence $\bar{T} = \lambda Id$. Note that for any orthonormal system of vectors $e_1, \dots, e_n \in V$

$$\sum \left\langle e_i, \bar{T}e_i\right\rangle = \sum \langle e_i, Te_i\rangle \leq 1,$$

that implies $\lambda n \leq 1$. Hence $\dim V \leq \frac{1}{\lambda}$. □

Corollary 2.5. *Every irreducible continuous representation of a compact group G is finite-dimensional.*

3. Orthogonality relations and Peter-Weyl Theorem

If $\rho : G \to \mathrm{GL}(V)$ is a unitary representation. Define a matrix coefficient by the formula

$$f_{v,w}(g) = \langle w, \rho_g v\rangle .$$

It is easy to check that

$$f_{v,w}\left(g^{-1}\right) = \bar{f}_{w,v}(g) \tag{3.1}$$

Theorem 3.1. *For an irreducible unitary representation $\rho : G \to \mathrm{GL}(V)$*

$$\langle f_{v,w}, f_{v',w'}\rangle = \int_G \bar{f}_{v,w}(g) f_{v',w'}(g)\, dg = \frac{1}{\dim \rho} \langle v, v'\rangle \langle w', w\rangle .$$

The matrix coefficient of two non-isomorphic representation are orthogonal in $L^2(G)$.

Proof. Define $T \in \mathrm{End}_{\mathbb{C}}(V)$

$$Tx = \langle v, x\rangle v'$$

and

$$\bar{T} = \int_G \rho_g T \rho_g^{-1} dg.$$

As follows from Shur's lemma, $\bar{T} = \lambda Id$. Since

$$\operatorname{tr} \bar{T} = \operatorname{tr} T = \langle v, v'\rangle ,$$

we obtain

$$\bar{T} = \frac{\langle v, v'\rangle}{\dim \rho}.$$

Hence

$$\left\langle w', \bar{T} w\right\rangle = \frac{1}{\dim \rho} \langle v, v'\rangle \langle w', w\rangle .$$

On the other hand,

$$\left\langle w', \bar{T} w\right\rangle = \int_G \left\langle w', \left\langle v, \rho_g^{-1} w\right\rangle \rho_g v'\right\rangle dg = \int_G f_{w,v}\left(g^{-1}\right) f_{v',w'}(g)\, dg =$$

$$= \int_G \bar{f}_{v,w}(g) f_{v',w'}(g)\, dg = \frac{1}{\dim \rho} \langle f_{v,w}, f_{v',w'}\rangle .$$

In $f_{v,w}$ and $f_{v',w'}$ are matrix coefficients of two non-isomorphic representation, the $\bar{T} = 0$, and the calculation is even simpler. □

Corollary 3.2. *Let ρ and σ be two irreducible representations, then $\langle \chi_\rho, \chi_\sigma\rangle = 1$ if ρ is isomorphic to σ and $\langle \chi_\rho, \chi_\sigma\rangle = 0$ otherwise.*

Theorem 3.3. *(Peter-Weyl) Matrix coefficient form a dense set in $L^2(G)$ for a compact group G.*

Proof. We will prove the Theorem under assumption that $G \subset \mathrm{GL}(E)$, in other words we assume that G has a faithful finite-dimensional representation. Let $M = \mathrm{End}_{\mathbb{C}}(E)$. The polynomial functions $\mathbb{C}[M]$ on M form a dense set in the space of continuous functions on G (Weierstrass theorem), and continuous functions are dense in $L^2(G)$. On the other hand, $\mathbb{C}[M]$ is spanned by matrix coefficients of all representations in $T(E) = \oplus_{n=0}^{\infty} E^{\otimes n}$. Hence matrix coefficients are dense in $L^2(G)$. □

Corollary 3.4. *The characters of irreducible representations form an orthonormal basis in the subspace of class function in* $L^2(G)$.

Corollary 3.5. *Let* G *be a compact group and* R *denote the representation of* $G \times G$ *in* $L^2(G)$ *given by the formula*

$$R_{s,t} f(x) = f\left(s^{-1} x t\right).$$

Then

$$L^2(G) \cong \oplus_{\rho \in \widehat{G}} V_\rho \boxtimes V_\rho^*,$$

where $\widehat{G}$ *denotes the set of isomorphism classes of irreducible unitary representations of* G *and the direct sum is in the sense of Hilbert spaces.*

Remark 3.6. Note that it follows from the proof of Theorem 3.3, that if E is a faithful representation of a compact group G, then all other irreducible representations appear in $T(E)$ as subrepresentations.

4. Examples

Example 1. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, $z = e^{i\theta}$. The invariant measure on S^1 is $\frac{d\theta}{2\pi}$ The irreducible representations are one dimensional. They are given by the characters $\chi_n : S^1 \to \mathbb{C}^*$, where $\chi_n(\theta) = e^{in\theta}$. Hence $\widehat{S}^1 = \mathbb{Z}$ and

$$L^2\left(S^1\right) = \oplus_{n \in \mathbb{Z}} \mathbb{C} e^{in\theta},$$

this is well-known fact that every periodic function can be extended in Fourier series.

Example 2. Let $G = SU_2$. Then G consists of all matrices

$$\begin{matrix} a & b \\ -\bar{b} & \bar{a}, \end{matrix}$$

satisfying the relations $|a|^2 + |b|^2 = 1$. One also can realize SU_2 as the subgroup of quaternions with norm 1. Thus, topologically SU_2 is isomorphic to the three-dimensional sphere S^3. To find all irreducible representation of SU_2 consider the polynomial ring $\mathbb{C}[x, y]$ with the action of SU_2 given by the formula

$$\rho_g(x) = ax + \mathrm{by},\ \rho_g(y) = -\bar{b}x + \bar{a}y, \text{ if } g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}.$$

Let ρ_n be the representation of G in the space $\mathbb{C}_n[x,y]$ of homogeneous polynomials of degree n. The monomials x^n, $x^{n-1}y, \dots, y^n$ form a basis of $\mathbb{C}_n[x,y]$. Therefore $\dim \rho_n = n+1$. We claim that all ρ_n are irreducible and that every irreducible representation of SU_2 is isomorphic to ρ_n. Hence $\widehat{G} = \mathbb{Z}_+$. We will show this by checking that the characters χ_n of ρ_n form an orthonormal basis in the Hilbert space of class functions on G.

Note that every unitary matrix is diagonal in some orthonormal basis, therefore every conjugacy class of SU_2 intersects the diagonal subgroup. Moreover, $\left(\begin{smallmatrix} z & 0 \\ 0 & \bar{z} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \bar{z} & 0 \\ 0 & z \end{smallmatrix}\right)$ are conjugate. Hence the set of conjugacy classes can be identified with S^1 quotient by the equivalence relation $z \sim \bar{z}$. Let $z = e^{i\theta}$, then

$$\chi_n(z) = z^n + z^{n-2} + \dots + z^{-n} = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}} = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{4.1}$$

Now let us calculate the scalar product in the space of class function. It is clear that the invariant measure dg on G is proportional to the standard volume form on the three-dimensional sphere induced by the volume form on $\mathbb{R}^4$. Let $C(\theta)$ denote the conjugacy class of all matrices with eigenvalues $e^{i\theta}, e^{-i\theta}$. The characteristic polynomial of a matrix from $C(\theta)$ equals $t^2 - 2\cos\theta t + 1$. Thus, we obtain $a + \bar{a} = 2\cos\theta$, or $a = \cos\theta + yi$ for real y. Hence $C(\theta)$ satisfy the equation

$$|a|^2 + |b^2| = \cos^2\theta + y^2 + |b|^2 = 1,$$

or

$$y^2 + |b|^2 = \sin^2\theta.$$

In other words, $C(\theta)$ is a two-dimensional sphere of radius $\sin\theta$. Hence for a class function ϕ on G

$$\int \phi(g)\, dg = \frac{1}{\pi}\int_0^{2\pi} \phi(\theta)\sin^2\theta d\theta.$$

All class function are even functions on S^1, i.e. they satisfy the condition $\phi(-\theta) = \phi(\theta)$. One can see easily from (4.1) that $\chi_n(\theta)$ form an orthonormal basis in the space of even function on the circle with respect to the Hermitian product

$$\langle \varphi, \eta \rangle = \frac{1}{\pi}\int_0^{2\pi} \bar{\varphi}(\theta)\,\eta(\theta)\sin^2\theta d\theta.$$

Example 3. Let $G = SO_3$. Recall that SU_2 can be realized as the set of quaternions with norm 1. Consider the representation γ of SU_2 in $\mathbb{H}$ defined by the formula $\gamma_g(\alpha) = g\alpha g^{-1}$. One can see that the 3-dimensional space $\mathbb{H}_{\text{im}}$ of pure imaginary quaternions is invariant and $(\alpha, \beta) = \operatorname{Re}\left(\alpha\bar{\beta}\right)$ is invariant positive definite scalar product on $\mathbb{H}_{\text{im}}$. Therefore ρ defines a homomorphism $\gamma\colon SU_2 \to SO_3$. Check that $\operatorname{Ker}\gamma = \{1, -1\}$ and that γ is surjetive. Hence $SO_3 \cong SU_2/\{1,-1\}$. Thus, every representation of SO_3 can be lifted to the representations of SU_2, and a representation of SU_2 factors to the representation of SO_3 iff it is trivial on -1. One can check easily that $\rho_n(-1) = 1$ iff n is even. Thus, an irreducible representations of SO_3 is isomorphic to ρ_{2m} and $\dim \rho_{2m} = 2m + 1$. Below we give an independent realization of irreducible representation of SO_3.

Harmonic analysis on a sphere. Consider the sphere S^2 in $\mathbb{R}^3$ defined by the equation $x^2 + y^2 + z^2 = 1$. It is clear that SO_3 acts in the space of complex-valued functions on S^2. Introduce differential operators in $\mathbb{R}^3$:

$$e = \frac{-1}{2}\left(x^2 + y^2 + z^2\right), \; h = x\partial_x + y\partial_y + z\partial_z + \frac{3}{2}, \; f = \frac{1}{2}\left(\partial_x^2 + \partial_y^2 + \partial_z^2\right),$$

note that e, f, and h commute with the action of SO_3 and satisfy the relations

$$[e, f] = h, \;\; [h, e] = 2e, \;\; [h, f] = -2f.$$

Let P_n be the space of homogeneous polynomial of degree n and $H_n = \operatorname{Ker} f \cap P_n$. The polynomials of H_n are harmonic polynomials since they are annihilated by Laplace operator. For any $\varphi \in P_n$

$$h(\varphi) = \left(n + \frac{3}{2}\right)\varphi.$$

If $\varphi \in H_n$, then

$$fe(\varphi) = ef(\varphi) - h(\varphi) = -\left(n + \frac{3}{2}\right)\varphi,$$

and by induction

$$fe^k(\varphi) = efe^{k-1}(\varphi) - he^{k-1}(\varphi) = -\left(nk + k(k-1) + \frac{3k}{2}\right)e^{k-1}\varphi.$$

In particular, this implies that

$$fe^k(H_n) = e^{k-1}(H_n). \tag{4.2}$$

We prove that

$$P_n = H_n \oplus e(H_{n-2}) \oplus e^2(H_{n-4}) + \dots \tag{4.3}$$

by induction on n. Indeed, by induction assumption

$$P_{n-2} = H_{n-2} \oplus e(H_{n-4}) + \dots,$$

then (4.2) implies $fe(P_{n-2}) = P_{n-2}$. Hence $H_n \cap eP_{n-2} = 0$. On the other hand, $f\colon P_n \to P_{n-2}$ is surjective, and therefore $\dim H_n + \dim P_{n-2} = \dim P_n$. Therefore

$$P_n = H_n \oplus P_{n-2}, \tag{4.4}$$

which implies (4.3). Note that after restriction on S^2, the operator e acts as the multiplication on $\frac{-1}{2}$.

Hence (4.3) implies that

$$\mathbb{C}\left[S^2\right] = \oplus_{n\geq 0} H_n.$$

To calculate the dimension of H_n use (4.4)

$$\dim H_n = \dim P_n - \dim P_{n-2} = \frac{(n+1)(n+2)}{2} - \frac{n(n-1)}{2} = 2n + 1.$$

Finally, we claim that the representation of SO_3 in H_n is irreducible and isomorphic to ρ_{2n}. Check that $\varphi = (x+iy)^n \in H_n$ and the rotation on the angle θ about $z-$ axis maps φ to $e^{in\theta}\varphi$. Since this rotation is the image of

$$\begin{matrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2}, \end{matrix}$$

under the homomorphism $\gamma : SU_2 \to SO_3$, the statement follows from (4.1).

Recall now the following theorem (Lecture Notes 1).

A convex centrally symmetric solid in $\mathbb{R}^3$ is uniquely determined by the areas of the plane cross-sections through the origin.

A convex solid B can be defined by an even continuous function on S^2. Indeed, for each unit vector v let

$$\varphi(v) = \sup\left\{t^2/2 \mid tv \in B\right\}.$$

Define a linear operator T in the space of all even continuous functions on S^2 by the formula

$$T\varphi(v) = \int_0^{2\pi} \varphi(w)\, d\theta,$$

where w runs the set of unit vectors orthogonal to v, and θ is the angular parameter on the circle $S^2 \cap v^\perp$. Check that $T\varphi(v)$ is the area of the cross section by the plane $v^\perp$. We have to prove that T is invertible.

Obviously T commutes with the SO_3-action. Therefore T can be diagonalized. Moreover, T acts on H_{2n} as the scalar operator $\lambda_n Id$. We have to check that $\lambda_n \neq 0$ for all n. Let $\varphi = (x+iy)^{2n} \in H_{2n}$. Then $\varphi(1,0,0) = 1$ and

$$T\varphi(1,0,0) = \int_0^{2\pi} (iy)^{2n}\, d\theta = (-1)^n \int_0^{2\pi} \sin^{2n}\theta d\theta,$$

here we take the integral over the circle $y^2 + z^2 = 1$, and assume $y = \sin\theta$, $z = \cos\theta$. Since $T\varphi = \lambda_n\varphi$, we obtain

$$\lambda_n = (-1)^n \int_0^{2\pi} \sin^{2n}\theta d\theta \neq 0.$$