

REPRESENTATION THEORY

WEEK 6

1. DIMENSION FORMULAE

For a Young tableau (diagram) λ , λ_i denotes the number of boxes in the i -th row. We write $\alpha \in \lambda$ if α is a box of λ . Let $|\lambda|$ denote the number of boxes in λ and $\lambda - \alpha$ denote a diagram which can be obtained from λ by removing one box. For example, for partition $\lambda = (5, 3, 1)$ the possible $\lambda - \alpha$ are $(4, 3, 1)$, $(5, 2, 1)$ and $(5, 3)$.

Theorem 1.1. $\text{Res}_{S_{n-1}} V_\lambda = \bigoplus V_{\lambda - \alpha}$.

Lemma 1.2. *Let $|\mu| = |\lambda| - 1$. If the diagram of μ is different from the diagram of $\lambda - \alpha$ for all possible α , then there are i, j either in the same row of λ and in the same column of μ or in the same row of μ and in the same column of λ .*

Proof. Let λ and μ do not satisfy the condition of Lemma. Choose the smallest k such that $\lambda_k \neq \mu_k$. Assume first, that $\mu_k > \lambda_k$, then by pigeon hole principle one can find two entries of k -th row of μ in the same column of λ . (We assume that this does not happen with the first $k - 1$ rows).

Assume now that $\lambda_k > \mu_k$. Since λ has just one more entry than μ , $\lambda_k > \mu_k + 1$ implies that two entries in the k -th row of λ are in the same column of μ . Therefore $\lambda_k = \mu_k + 1$, moreover the last entry appears in the first k rows of λ . In this case we move to the next row, and step by step prove that $\lambda_i = \mu_i$ for all $i \neq k$. Hence $\mu = \lambda - \alpha$. □

Lemma 1.3. *If the diagram of μ is different from the diagram of $\lambda - \alpha$ for all possible α , then $c_\mu \mathcal{A} c_\lambda = 0$.*

Proof. Let $s \in S_n$. First assume that there are two entries in the same row of μ and in the same column of λ . Then the same is true for μ and $s(\lambda)$ for any $s \in S_n$ as we can see from the proof of Lemma 1.2. Hence for this pair of entries i, j we have

$$a_\mu (ij)^2 b_{s(\lambda)} = a_\mu b_{s(\lambda)} = -a_\mu b_{s(\lambda)} = 0.$$

Therefore $a_\mu s b_\lambda s^{-1} = 0$ and $a_\mu s b_\lambda = 0$ for any $s \in S_n$. That implies $a_\mu \mathcal{A} b_\lambda = 0$. Similarly we can prove that if there are two entries in the same column of μ and in the same row of λ , then $b_\mu \mathcal{A} a_\lambda = 0$. Together that implies $c_\mu \mathcal{A} c_\lambda = 0$. □

Corollary 1.4. *If a Young diagram μ can not be obtained from λ by removing one box, then the multiplicity of V_μ in $\text{Res}_{S_{n-1}} V_\lambda$ is zero.*

Proof. Follows from Lemma 5.12 (lecture notes 5). \square

Lemma 1.5. *Let $\mu = \lambda - \alpha$. Then $c_\mu c_\lambda \neq 0$. Therefore $f : V_\mu \rightarrow V_\lambda$ given by $f(x) = xc_\lambda$ is an injective homomorphism of S_{n-1} -modules.*

Proof. If we write

$$c_\mu c_\lambda = \sum_{s \in S_n} u_s s, \quad c_\mu^2 = \sum_{s \in S_{n-1}} v_s s,$$

then as one can easily see that $u_s = v_s$ for any $s \in S_{n-1}$. \square

Lemma 1.6. *If $\mu = \lambda - \alpha$, then $c_\mu \mathcal{A}c_\lambda \subset \mathbb{Q}c_\mu c_\lambda$.*

Proof. First, prove that if there is no $(ij) \in Q_\mu P_{s(\lambda)}$ then $s \in Q_\lambda P_\lambda$. Thus, $b_\mu s a_\mu \neq 0$, or equivalently $b_\mu a_{s(\lambda)} \neq 0$, implies $s \in Q_\lambda P_\lambda$. Then prove that for any $s \in Q_\lambda P_\lambda$, $c_\mu s c_\lambda \in \mathbb{Q}c_\mu c_\lambda$. \square

Corollary 1.4, Lemma 1.5 and Lemma 1.6 imply Theorem 1.1.

Corollary 1.7.

$$\dim V_\lambda = \sum \dim V_{\lambda - \alpha}.$$

Remark 1.8. Every function $f(\lambda)$ on the set of Young diagrams satisfying

$$(1.1) \quad f(\lambda) = \sum f(\lambda - \alpha), \quad f(1) = 1$$

coincides with $\dim V_\lambda$.

Corollary 1.9. $\text{Ind}_{S_n}^{S_{n+1}} V_\lambda = \bigoplus V_\mu$, where μ runs the set of all diagrams obtained from λ by adding one box.

A Young tableau is *standard* if entries in every row and entries in every column are in increasing order.

Corollary 1.10. $\dim V_\lambda$ equals the number of all standard tableaux on a diagram λ .

Proof. Check that the number d_λ of standard tableaux satisfies (1.1). \square

For a box $\alpha \in \lambda$, let h_α be the hook diagram containing α , all boxes below α and all boxes to the right of α . Let

$$h(\lambda) = \prod_{\alpha \in \lambda} |h_\alpha|.$$

Example. If λ is $(3,2,1)$, then $h(\lambda) = 45$.

Lemma 1.11. *Let $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\bar{\lambda} = (\lambda_1 + k - 1, \lambda_2 + k - 2, \dots, \lambda_k)$. Then*

$$(1.2) \quad h(\lambda) = \frac{\bar{\lambda}_1! \dots \bar{\lambda}_k!}{\prod_{i < j} (\bar{\lambda}_i - \bar{\lambda}_j)}.$$

Proof. Direct calculation. □

Lemma 1.12. *Let*

$$V(x_1, \dots, x_k) = \prod_{i < j} (x_i - x_j).$$

Then

$$(1.3) \quad \sum_{i=1}^k x_i (V(x_1, \dots, x_k) - V(x_1, \dots, x_i - 1, \dots, x_k)) = \frac{k(k-1)}{2} V(x_1, \dots, x_k).$$

Proof. Since V is a skew symmetric polynomial of x , it is easy to check that $(x_i - x_j)$ divides the left hand side of the identity. Since the degree of the left hand side polynomial is $\frac{k(k-1)}{2}$, the same as the degree of V , the LHS polynomial is proportional to V . The leading coefficient of LHS is the same as of

$$\sum_{i=1}^k x_i \frac{\partial}{\partial x_i} V(x_1, \dots, x_k) = \frac{k(k-1)}{2} V(x_1, \dots, x_k).$$

That proves the identity. □

Lemma 1.13.

$$\frac{n}{h(\lambda)} = \sum \frac{1}{h(\lambda - \alpha)}.$$

Proof. Using (1.2) write

$$n \frac{V(\bar{\lambda}_1, \dots, \bar{\lambda}_k)}{\bar{\lambda}_1! \dots \bar{\lambda}_k!} = \sum_{i=1}^k \frac{V(\bar{\lambda}_1, \dots, \bar{\lambda}_i - 1, \dots, \bar{\lambda}_k)}{\bar{\lambda}_1! \dots (\bar{\lambda}_i - 1)! \dots \bar{\lambda}_k!}.$$

This is equivalent to

$$(1.4) \quad nV(\bar{\lambda}_1, \dots, \bar{\lambda}_k) = \sum_{i=1}^k \bar{\lambda}_i V(\bar{\lambda}_1, \dots, \bar{\lambda}_i - 1, \dots, \bar{\lambda}_k).$$

Use now that $\bar{\lambda}_1 + \dots + \bar{\lambda}_k = n + \frac{k(k-1)}{2}$ and apply (1.3) to prove (1.4). □

Corollary 1.14. (*Hook formula*) $\dim V_\lambda = \frac{|\lambda|!}{h(\lambda)}.$

Proof. Just check that $\frac{|\lambda|!}{h(\lambda)}$ satisfies (1.1). □

2. REPRESENTATIONS OF GL_k .

Matrix coefficients. Let $\rho : G \rightarrow GL(V)$ be a (finite-dimensional) representation. For any $\varphi \in V^*, v \in V$ define

$$f_{v,\varphi}(s) = \langle \varphi, \rho_s v \rangle.$$

This function f is called a *matrix coefficient*.

Let $G = GL_k = GL(\mathbb{C}^k)$ and $\mathbb{C}[G]$ denote the space of all polynomial functions on G . We call a representation $\rho : G \rightarrow GL(V)$ *polynomial* if $f_{v,\varphi} \in \mathbb{C}[G]$ for all $v \in V, \varphi \in V^*$.

Examples. The standard representation in the space $E = \mathbb{C}^k$ is polynomial, but the dual representation in E^* is not.

The whole space $\mathbb{C}[G]$ has a natural structure of a representation if we put

$$R_g f(x) = f(xg).$$

Check that the space $\mathbb{C}_n[G]$ of homogeneous polynomials of degree n is invariant. Thus, there is a decomposition

$$\mathbb{C}[G] = \bigoplus_{n=0}^{\infty} \mathbb{C}_n[G].$$

Let $\rho : G \rightarrow GL(V)$ be a polynomial representation. For any $\varphi \in V^*$ define a map $\rho'_\varphi : V \rightarrow \mathbb{C}[G]$ by the formula

$$\rho'_\varphi(v) = f_{v,\varphi}.$$

Check that this map is an intertwiner, i.e.

$$\rho'_\varphi(\rho_g v) = R_g \rho'_\varphi(v).$$

That implies

Lemma 2.1. *Every irreducible polynomial representation of G is a subrepresentation in $\mathbb{C}[G]$.*

Lemma 2.2. *Consider the representation of G in $(E^*)^{\otimes n} \otimes E^{\otimes n}$ given by the formula*

$$\rho_g(\varphi_1 \otimes \cdots \otimes \varphi_n \otimes v_1 \otimes \cdots \otimes v_n) = \varphi_1 \otimes \cdots \otimes \varphi_n \otimes gv_1 \otimes \cdots \otimes gv_n,$$

for all $\varphi_i \in E^*, v_j \in E$.

The map $\pi : (E^*)^{\otimes n} \otimes E^{\otimes n} \rightarrow \mathbb{C}_n[G]$ given by

$$\varphi \otimes v \mapsto f_{v,\varphi}$$

for each $\varphi \in (E^*)^{\otimes n}, v \in E^{\otimes n}$, is surjective.

Proof. Let e_1, \dots, e_k be a basis in E and f_1, \dots, f_k be the dual basis in E^* . Then

$$f_{j_1} \otimes \cdots \otimes f_{j_n} \otimes e_{i_1} \otimes \cdots \otimes e_{i_n} \mapsto g_{i_1 j_1} \cdots g_{i_n j_n},$$

where g_{ij} is a matrix entry of a matrix g in the basis e_1, \dots, e_k . Thus, the monomial basis of $\mathbb{C}_n[G]$ belongs to the image of π . \square

Remark 2.3. Here I made a mistake during the lecture. To get an isomorphism we have to consider $S^n(E^* \otimes E) \subset (E^*)^{\otimes n} \otimes E^{\otimes n}$.

To classify polynomial irreducible representation of G we have to find all irreducible subrepresentations of $E^{\otimes n}$. We will do this in the next section.

3. DUALITY BETWEEN GL_k AND S_n

Consider the representation $\rho : S_n \rightarrow GL(E^{\otimes n})$ defined by the formula

$$\rho_s(v_1 \otimes \cdots \otimes v_n) = v_{s(1)} \otimes \cdots \otimes v_{s(n)},$$

and the representation $\rho : GL_k \rightarrow GL(E^{\otimes n})$ defined by

$$\rho_g(v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n.$$

We see immediately that $\rho_s \circ \rho_g = \rho_g \circ \rho_s$ for any $s \in S_n, g \in GL_k$. Thus we can consider ρ as the representation of the direct product $S_n \times GL_k$.

Theorem 3.1. *Let $\Gamma_{n,k}$ denote the set of all Young diagrams with n boxes such that the number of rows is not bigger than k . Then*

$$E^{\otimes n} = \bigoplus_{\lambda \in \Gamma_{n,k}} V_\lambda \boxtimes W_\lambda,$$

where V_λ is the irreducible representation of S_n associated with λ and W_λ is an irreducible representation of GL_k . Moreover, W_λ and W_μ are not isomorphic if $\lambda \neq \mu$.

Corollary 3.2. *Fill the boxes of λ in some way. Then $\rho_{c_\lambda}(E^{\otimes n})$ is an invariant subspace isomorphic to W_λ .*

Example. Let $\lambda = (n)$ be a one row diagram. Then V_λ is the trivial representation of S_n , $c_\lambda = \sum_{s \in S_n} s$, and $W_\lambda = S^n(E)$.

If $k > n$, and $\lambda = (1, \dots, 1)$ (one row), then V_λ is the sign representation, $c_\lambda = \sum_{s \in S_n} (-1)^s s$, and $W_\lambda = \Lambda^n(E)$.

To prove Theorem 3.1 we need the following general statement.

Theorem 3.3. *Let $\rho : G \rightarrow GL(V), \sigma : K \rightarrow GL(V)$ be two representations in the same vector space V over algebraically closed F . Let*

$$\text{End}_G(V) = \sigma(F(K))$$

and ρ is completely reducible. Then

$$V = \bigoplus_{i=1}^m V_i \boxtimes W_i,$$

where V_i is an irreducible representation of G , W_i is an irreducible representation of K . Moreover, V_i is not isomorphic to V_j if $i \neq j$ and similarly, W_i is not isomorphic to W_j if $i \neq j$.

Proof. Since ρ is completely reducible, one can write

$$V = \bigoplus_{i=1}^m (V_i \otimes W_i),$$

the action of G is trivial on W_i . Then

$$\text{End}_G(V) = \prod_{i=1}^m \text{End}_F(W_i).$$

Thus, $\sigma: F(K) \rightarrow \text{End}_F(W_i)$ is surjective, that implies that each W_i is irreducible over K and $W_i \not\cong W_j$ if $i \neq j$. \square

Remark 3.4. In general, we say that G and K satisfying the conditions of Theorem 3.3 form a *dual pair*. Such situation often happens in representation theory. The simplest example is an action of $G \times G$ in $k(G)$ given by

$$R_{(g,h)} \sum_{s \in G} u_s s = \sum_{s \in G} u_s g s h^{-1}.$$

Lemma 3.5. *In the situation of Theorem 3.1 we have*

$$\text{End}_{S_n}(E^{\otimes n}) = \rho(\mathbb{C}(\text{GL}_k)).$$

Proof. Let M_k denote the algebra $\text{End}_{\mathbb{C}}(E)$, in other words, M_k is the matrix algebra. First,

$$\text{End}_{\mathbb{C}}(E^{\otimes n}) = M_k^{\otimes n}.$$

Thus, we are looking at the S_n invariant subalgebra

$$\text{End}_{S_n}(E^{\otimes n}) = (M_k^{\otimes n})^{S_n} = S^n(M_k).$$

In other words, $\text{End}_{\mathbb{C}}(E^{\otimes n})$ is spanned by

$$\sum_{s \in S_n} m_{s(1)} \otimes \cdots \otimes m_{s(n)}$$

for all possible $m_1, \dots, m_n \in M_k$.

Our next claim is that $\text{End}_{\mathbb{C}}(E^{\otimes n})$ is the span of $m \otimes \cdots \otimes m$ for all possible $m \in M_k$. It follows from the following

Lemma 3.6. *For an arbitrary vector space V , $S^n(V)$ is spanned by v^n for all $v \in V$.*

Proof. Everybody knows the formula

$$4xy = (x+y)^2 - (x-y)^2,$$

which proves the statement for $n = 2$. Less known is the following general formula

$$2^n x_1 \cdots x_n = \sum_{i_2=0,1, \dots, i_n=0,1} (-1)^{i_2+\dots+i_n} \left(x_1 + (-1)^{i_2} x_2 + \cdots + (-1)^{i_n} x_n \right)^n.$$

\square

Now let U be the span of $g \otimes \cdots \otimes g$ ($g \in \mathrm{GL}_k$) in $\mathrm{End}_{\mathbb{C}}(E^{\otimes n}) = S^n(M_k)$. By definition, $U = \rho(\mathbb{C}(\mathrm{GL}_k))$. Note that GL_k is a dense subset in M_k , therefore U is a dense subset in $S^n(M_k)$. But U is a linear subspace in $S^n(M_k)$. Hence $U = S^n(M_k) = \mathrm{End}_{\mathbb{C}}(E^{\otimes n})$. \square

Note that Lemma 3.5 together with Theorem 3.3 imply that

$$E^{\otimes n} = \bigoplus_{\lambda \in \Gamma'} V_{\lambda} \boxtimes W_{\lambda},$$

for some set Γ' of Young diagrams with n boxes. It is left to show that $\Gamma' = \Gamma_{n,k}$. Obviously, Γ' consists of all diagrams λ for which $W_{\lambda} = c_{\lambda}(E^{\otimes n}) \neq 0$. Fill the boxes of λ in increasing order from 1 to n from left to right starting from the top and consider c_{λ} defined by this tableaux. An element $v = e_{i_1} \otimes \cdots \otimes e_{i_n}$ of the standard basis in $E^{\otimes n}$ can be represented by the same tableau with entries e_{i_1}, \dots, e_{i_n} . If λ has more than k rows, then one can find e_j which appears twice in the same column. Then $b_{\lambda}(v) = 0$, and therefore $c_{\lambda}v = 0$. Since this holds for any basis vector, we have $c_{\lambda}(E^{\otimes n}) = 0$. Hence $\lambda \notin \Gamma'$ if λ has more than k rows. On the other hand, if λ has k or less rows, one can check that

$$c_{\lambda} \left(e_1^{\otimes \lambda_1} \otimes \cdots \otimes e_k^{\otimes \lambda_k} \right) \neq 0.$$

Therefore $\Gamma' = \Gamma_{n,k}$. Theorem 3.1 is proven.