REPRESENTATION THEORY.
WEEK 4

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1. Induced modules

Let $B \subset A$ be rings and M be a B-module. Then one can construct *induced* module $\mathrm{Ind}_B^A M = A \otimes_B M$ as the quotient of a free abelian group with generators from $A \times M$ by relations

$$(a_1 + a_2) \times m - a_1 \times m - a_2 \times m,\ a \times (m_1 + m_2) - a \times m_1 - a \times m_2,\ ab \times m - a \times bm,$$

and A acts on $A \otimes_B M$ by left multiplication. Note that $j : M \to A \otimes_B M$ defined by

$$j(m) = 1 \otimes m$$

is a homomorphism of B-modules.

Lemma 1.1. *Let N be an A-module, then for $\varphi \in \mathrm{Hom}_B(M, N)$ there exists a unique $\psi \in \mathrm{Hom}_A(A \otimes_B M, N)$ such that $\psi \circ j = \varphi$.*

Proof. Clearly, ψ must satisfy the relation

$$\psi(a \otimes m) = a\psi(1 \otimes m) = a\varphi(m).$$

It is trivial to check that ψ is well defined. □

Exercise. Prove that for any B-module M there exists a unique A-module satisfying the conditions of Lemma 1.1.

Corollary 1.2. *(Frobenius reciprocity.) For any B-module M and A-module N there is an isomorphism of abelian groups*

$$\mathrm{Hom}_B(M, N) \cong \mathrm{Hom}_A(A \otimes_B M, N).$$

Example. Let $k \subset F$ be a field extension. Then induction Ind_k^F is an exact functor from the category of vector spaces over k to the category of vector spaces over F, in the sense that the short exact sequence

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

becomes an exact sequence

$$0 \to F \otimes_k V_1 \otimes \to F \otimes_k V_2 \to F \otimes_k V_3 \to\ 0.$$

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In general, the latter property is not true. It is not difficult to see that induction is right exact, i.e. an exact sequence of B-modules

$$M \to N \to 0$$

induces an exact sequence of A-modules

$$A \otimes_B M \to A \otimes_B N \to \; 0.$$

But an exact sequence

$$0 \to M \to N$$

is not necessarily exact after induction.

Later we discuss general properties of induction but now we are going to study induction for the case of groups.

2. Induced representations for groups.

Let H be a subgroup of G and $\rho : H \to \mathrm{GL}(V)$ be a representation. Then the induced representation $\mathrm{Ind}_H^G \rho$ is by definition a $k(G)$-module

$$k(G) \otimes_{k(H)} V.$$

Lemma 2.1. *The dimension of $\mathrm{Ind}_H^G \rho$ equals the product of $\dim \rho$ and the index $[G : H]$ of H. More precisely, let S is a set of representatives of left cosets, i.e.*

$$G = \coprod_{s \in S} sH,$$

then

$$k(G) \otimes_{k(H)} V = \oplus_{s \in S} s \otimes V. \tag{2.1}$$

For any $t \in G$, $s \in S$ there exist unique $s' \in S$, $h \in H$ such that $ts = s'h$ and the action of t is given by

$$t(s \otimes v) = s' \otimes \rho_h v. \tag{2.2}$$

Proof. Straightforward check. □

Lemma 2.2. *Let $\chi = \chi_\rho$ and $\mathrm{Ind}_H^G \chi$ denote the character of $\mathrm{Ind}_H^G \rho$. Then*

$$\mathrm{Ind}_H^G \chi(t) = \sum_{s \in S, s^{-1}ts \in H} \chi\left(s^{-1}ts\right) = \frac{1}{|H|} \sum_{s \in G, s^{-1}ts \in H} \chi\left(s^{-1}ts\right). \tag{2.3}$$

Proof. (2.1) and (2.2) imply

$$\mathrm{Ind}_H^G \chi(t) = \sum_{s \in S, s' = s} \chi(h).$$

Since $s = s'$ implies $h = s^{-1}ts \in H$, we obtain the formula for the induced character. Note also that $\chi(s^{-1}ts)$ does not depend on a choice of s in a left coset. □

Corollary 2.3. *Let H be a normal subgroup in G. Then $\operatorname{Ind}_H^G \chi(t) = 0$ for any $t \notin H$.*

Theorem 2.4. *For any $\rho : G \to \mathrm{GL}(V)$, $\sigma\colon H \to \mathrm{GL}(W)$, we have the identity*

$$\left(\operatorname{Ind}_H^G \chi_\sigma, \chi_\rho\right)_G = \left(\chi_\sigma, \operatorname{Res}_H \chi_\rho\right)_H . \tag{2.4}$$

Here a subindex indicates the group where we take inner product.

Proof. It follows from Frobenius reciprocity, since

$$\dim \operatorname{Hom}_G \left(\operatorname{Ind}_H^G W, V\right) = \dim \operatorname{Hom}_H (W, V) .$$

□

Note that (2.4) can be proved directly from (2.3). Define two maps

$$\operatorname{Res}_H : \mathcal{C}(G) \to \mathcal{C}(H), \ \operatorname{Ind}_H^G : \mathcal{C}(H) \to \mathcal{C}(G),$$

the former is the restriction on a subgroup, the latter is defined by (2.3). Then for any $f \in \mathcal{C}(G), g \in \mathcal{C}(H)$

$$\left(\operatorname{Ind}_H^G g, f\right)_G = (g, \operatorname{Res}_H f)_H . \tag{2.5}$$

Example 1. Let ρ be a trivial representation of H. Then $\operatorname{Ind}_H^G \rho$ is the permutation representation of G obtained from the natural left action of G on G/H (the set of left cosets).

Example 2. Let $G = S_3$, $H = A_3$, ρ be a non-trivial one dimensional representation of H (one of two possible). Then

$$\operatorname{Ind}_H^G \chi_\rho (1) = 2, \ \operatorname{Ind}_H^G \chi_\rho (12) = 0, \ \operatorname{Ind}_H^G \chi_\rho (123) = -1.$$

Thus, by induction we obtain an irreducible two-dimensional representation of G.

Now consider another subgroup K of $G = S_3$ generated by the transposition (12), and let σ be the (unique) non-trivial one-dimensional representation of K. Then

$$\operatorname{Ind}_K^G \chi_\sigma (1) = 3, \ \operatorname{Ind}_K^G \chi_\sigma (12) = -1, \ \operatorname{Ind}_H^G \chi_\rho (123) = 0.$$

3. Double cosets and restriction to a subgroup

If K and H are subgroups of G one can define the equivalence relation on G: $s \sim t$ iff $s \in KtH$. The equivalence classes are called *double cosets*. We can choose a set of representative $T \subset G$ such that

$$G = \coprod_{s \in T} K \, \mathrm{tH} .$$

We define the set of double cosets by $K \backslash G / H$. One can identify $K \backslash G / H$ with K-orbits on $S = G/H$ in the obvious way and with G-orbits on $G/K \times G/H$ by the formula

$$KtH \to G(K, tH) .$$

Example. Let $\mathbb{F}_q$ be a field of q elements and $G = \mathrm{GL}_2(\mathbb{F}_q) \overset{\mathrm{def}}{=} \mathrm{GL}\left(\mathbb{F}_q^2\right)$. Let B be the subgroup of upper-triangular matrices in G. Check that $|G| = (q^2 - 1)(q^2 - q)$, $|B| = (q-1)^2 q$ and therefore $[G : B] = q + 1$. Identify G/B with the set of lines $\mathbb{P}^1$ in $\mathbb{F}_q^2$ and $B\backslash G/B$ with G-orbits on $\mathbb{P}^1 \times \mathbb{P}^1$. Check that G has only two orbits on $\mathbb{P}^1 \times \mathbb{P}^1$: the diagonal and its complement. Thus, $|B\backslash G/B| = 2$ and

$$G = B \cup BsB,$$

where

$$s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Theorem 3.1. *Let $T \subset G$ such that $G = \coprod_{s \in T} KtH$. Then*

$$\mathrm{Res}_K \mathrm{Ind}_H^G \rho = \oplus_{s \in T} \mathrm{Ind}_{K \cap sHs^{-1}}^K \rho^s,$$

where

$$\rho_h^s \overset{def}{=} \rho_{s^{-1}hs},$$

for any $h \in sHs^{-1}$.

Proof. Let $s \in T$ and $W^s = k(K)(s \otimes V)$. Then by construction, W^s is K-invariant and

$$k(G) \otimes_{k(H)} V = \oplus_{s \in T} W^s.$$

Thus, we need to check that the representation of K in W^s is isomorphic to $\mathrm{Ind}_{K \cap \mathrm{sHs}^{-1}}^K \rho^s$. We define a homomorphism

$$\alpha : \mathrm{Ind}_{K \cap sHs^{-1}}^K V \to W^s$$

by $\alpha(t \otimes v) = ts \otimes v$ for any $t \in K, v \in V$. It is well defined

$$\alpha(th \otimes v - t \otimes \rho_h^s v) = ths \otimes v - ts \otimes \rho_{s^{-1}hs} v = ts\left(s^{-1}hs\right) \otimes v - ts \otimes \rho_{s^{-1}hs} v = 0$$

and obviously surjective. Injectivity can be proved by counting dimensions. □

Example. Let us go back to our example $B \subset \mathrm{SL}_2(\mathbb{F}_q)$. Theorem 3.1 tells us that for any representation ρ of B

$$\mathrm{Ind}_B^G \rho = \rho \oplus \mathrm{Ind}_H^G \rho',$$

where $H = B \cap sBs^{-1}$ is a subgroup of diagonal matrices and

$$\rho' \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \rho \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$$

Corollary 3.2. *If H is a normal subgroup of G, then*

$$\mathrm{Res}_H \mathrm{Ind}_H^G \rho = \oplus_{s \in G/H} \rho^s.$$

4. Mackey's criterion

To find $\left(\operatorname{Ind}_H^G \chi, \operatorname{Ind}_H^G \chi\right)$ we can use Frobenius reciprocity and Theorem 3.1.

$$\left(\operatorname{Ind}_H^G \chi, \operatorname{Ind}_H^G \chi\right)_G = \left(\operatorname{Res}_H \operatorname{Ind}_H^G \chi, \chi\right)_H = \sum_{s\in T} \left(\operatorname{Ind}_{H\cap sHs^{-1}}^H \chi^s, \chi\right)_H =$$

$$= \sum_{s\in T} \left(\chi^s, \operatorname{Res}_{H\cap sHs^{-1}} \chi\right)_{H\cap sHs^{-1}} = (\chi,\chi)_H + \sum_{s\in T\setminus\{1\}} \left(\chi^s, \operatorname{Res}_{H\cap sHs^{-1}} \chi\right)_{H\cap sHs^{-1}}.$$

We call two representation *disjoint* if they do not have the same irreducible component, i.e. their characters are orthogonal.

Theorem 4.1. *(Mackey's criterion)* $\operatorname{Ind}_H^G \rho$ *is irreducible iff* ρ *is irreducible and* ρ^s *and* ρ *are disjoint representations of* $H \cap sHs^{-1}$ *for any* $s \in T\setminus\{1\}$.

Proof. Write the condition

$$\left(\operatorname{Ind}_H^G \chi, \operatorname{Ind}_H^G \chi\right)_G = 1$$

and use the above formula. □

Corollary 4.2. *Let* H *be a normal subgroup of* G. *Then* $\operatorname{Ind}_H^G \rho$ *is irreducible iff* ρ^s *is not isomorphic to* ρ *for any* $s \in G/H$, $s \notin H$.

Remark 4.3. Note that if H is normal, then G/H acts on the set of representations of H. In fact, this is a part of the action of the group $\operatorname{Aut} H$ of automorphisms of H on the set of representation of H. Indeed, if $\varphi \in \operatorname{Aut} H$ and $\rho : H \to \mathrm{GL}(V)$ is a representation, then $\rho^\varphi : H \to \mathrm{GL}(V)$ defined by

$$\rho_t^\varphi = \rho_{\varphi(t)},$$

is a new representation of H.

5. Some examples

Let H be a subgroup of G of index 2. Then H is normal and $G = H \cup sH$ for some $s \in G\setminus H$. Suppose that ρ is an irreducible representation of H. There are two possibilities

(1) ρ^s is isomorphic to ρ;
(2) ρ^s is not isomorphic to ρ.

Hence there are two possibilities for $\operatorname{Ind}_H^G \rho$:

(1) $\operatorname{Ind}_H^G \rho = \sigma \oplus \sigma'$, where σ and σ' are two non-isomorphic irreducible representations of G;
(2) $\operatorname{Ind}_H^G \rho$ is irreducible.

For instance, let $G = S_5$, $H = A_5$ and $\rho_1, \dots, \rho_5$ be irreducible representation of H, where the numeration is from lecture notes week 3. Then for $i = 1, 2, 3$

$$\operatorname{Ind}_H^G \rho_i = \sigma_i \oplus (\sigma_i \otimes \operatorname{sgn}),$$

here sgn denotes the sign representation. Furthermore, $\operatorname{Ind}_H^G \rho_4 \cong \operatorname{Ind}_H^G \rho_5$ is irreducible. Thus S_5 has two $1, 5, 4$-dimensional irreducible representations and one 6-dimensional.

Now let G be a subgroup of $\mathrm{GL}_2(\mathbb{F}_q)$ of matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

We want to classify irreducible representations of G over $\mathbb{C}$. $|G| = q^2 - q$, G has the following conjugacy classes

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix},$$

in the last case $a \neq 1$. Note that the subgroup H of matrices

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

is normal, $G/H \cong \mathbb{F}_q^* \cong Z_{q-1}$. Therefore G has $q-1$ one-dimensional representations which can be lifted from G/H. That leaves one more representation, its dimension must be $q-1$. We hope to obtain it by induction from H. Let σ be a non-trivial irreducible representation of H (one-dimensional). Then $\dim \operatorname{Ind}_H^G \sigma = q - 1$ as required. Note that for any previously constructed one-dimensional representation ρ of G we have

$$\left(\operatorname{Ind}_H^G \sigma, \rho\right)_G = (\sigma, \operatorname{Res}_H \rho)_H = 0,$$

as $\operatorname{Res}_H \rho$ is trivial. Therefore $\operatorname{Ind}_H^G \sigma$ is irreducible. The character takes values $q-1$, -1 and 0 on the corresponding conjugacy classes.

Remark 5.1. To find all one-dimensional representation of a group G, find its commutator G', which is a subgroup generated by $ghg^{-1}h^{-1}$ for all $g, h \in G$. One-dimensional representations of G are lifted from one-dimensional representations of G/G'.