REPRESENTATION THEORY.
WEEK 3

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1. Examples.

Example 1. Let $G = S_3$. There are three conjugacy classes in G, which we denote by some element in a class:1,(12),(123). Therefore there are three irreducible representations, denote their characters by χ_1, χ_2 and χ_3. It is not difficult to see that we have the following table of characters

	1	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

The characters of one-dimensional representations are given in the first and the second row, the last character χ_3 can be obtained by using the identity

$$\chi_{\text{perm}} = \chi_1 + \chi_3, \tag{1.1}$$

where χ_{perm} stands for the character of the permutation representation.

Example 2. Let $G = S_4$. In this case we have the following character table (in the first row we write the number of elements in each conjugacy class).

	1	6	8	3	6
	1	(12)	(123)	(12) (34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	3	1	0	-1	-1
χ_4	3	-1	0	-1	1
χ_5	2	0	-1	2	0

First two rows are characters of one-dimensional representations. The third can be obtained again from (1.1), $\chi_4 = \chi_2\chi_3$, the corresponding representation is obtained as the tensor product $\rho_4 = \rho_2 \otimes \rho_3$. The last character can be found from the orthogonality relation. Alternative way to describe ρ_5 is to consider S_4/V_4, where

$$V_4 = \{1, (12)\,(34)\,, (13)\,(24)\,, (14)\,(23)\}$$

Date: September 16, 2005.

is the Klein subgroup. Observe that $S_4/V_4 \cong S_3$, and therefore the two-dimensional representation σ of S_3 can be extended to the representation of S_4 by the formula

$$\rho_5 = \sigma \circ p,$$

where $p : S_4 \to S_3$ is the natural projection.

Solution of the marcian problem. Recall that S_4 is isomorphic to the group of rotations of a cube. Hence it acts on the set of faces F, and therefore we have a representation

$$\rho\colon S_4 \to \mathrm{GL}\left(\mathbb{C}\left(F\right)\right).$$

It is not difficult to calculate χ_ρ using the formula

$$\chi_\rho\left(s\right) = \left|\left\{x \in F \mid s\left(x\right) = x\right\}\right|.$$

We obtain

$$\chi_\rho\left(1\right) = 6,\ \chi_\rho\left(\left(12\right)\right) = \chi_\rho\left(\left(123\right)\right) = 0,\ \chi_\rho\left(\left(12\right)\left(34\right)\right) = \chi_\rho\left(\left(1234\right)\right) = 2.$$

Furthermore,

$$(\chi_\rho, \chi_1) = (\chi_\rho, \chi_4) = (\chi_\rho, \chi_5) = 1,\ \ (\chi_\rho, \chi_2) = (\chi_\rho, \chi_3) = 0.$$

Hence $\chi_\rho = \chi_1 + \chi_4 + \chi_5$, and $\mathbb{C}\left(F\right) = W_1 \oplus W_2 \oplus W_3$ the sum of three invariant subspaces. The intertwining operator $T : \mathbb{C}\left(F\right) \to \mathbb{C}\left(F\right)$ of food redistribution must be a scalar operator on each W_i by Schur's Lemma. Note that

$$W_1 = \left\{\sum_{x\in F} ax \mid a \in \mathbb{C}\right\},$$

$$W_2 = \left\{\sum_{x\in F} a_x x \mid a_x = -a_{x_{\mathrm{op}}}\right\},$$

$$W_3 = \left\{\sum_{x\in F} a_x x \mid \sum a_x = 0, a_x = a_{x_{\mathrm{op}}}\right\},$$

where x_{op} denotes the face opposite to the face x. A simple calculation shows that $T_{|W_1} = \mathrm{Id}$, $T_{|W_2} = 0$, $T_{|W_3} = -\frac{1}{2}Id$. Therefore $T^n\left(v\right)$ approaches a vector in W_1 as $n \to \infty$, and eventually everybody will have the same amount of food.

Example 3. Now let $G = A_5$. There are 5 irreducible representation of G over $\mathbb{C}$. Here is the character table

	1	20	15	12	12
	1	(123)	(12) (34)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	4	1	0	-1	-1
χ_3	5	-1	1	0	0
χ_4	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_5	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

To obtain χ_2 use the permutation representation and (1.1) again. Let χ_{sym} and χ_{alt} be the characters of the second symmetric and the second exterior powers of ρ_2 respectively. Then we obtain

	1	(123)	(12) (34)	(12345)	(12354)
χ_{sym}	10	1	2	0	0
χ_{alt}	6	0	−2	1	1

It is easy to check that

$$(\chi_{\text{sym}}, \chi_{\text{sym}}) = 3,\ (\chi_{\text{sym}}, \chi_1) = (\chi_{\text{sym}}, \chi_2) = 1.$$

Therefore

$$\chi_3 = \chi_{\text{sym}} - \chi_1 - \chi_2$$

is the character of an irreducible representation.

Furthermore,

$$(\chi_{\text{alt}}, \chi_{\text{alt}}) = 2,\ (\chi_{\text{alt}}, \chi_1) = (\chi_{\text{alt}}, \chi_2) = (\chi_{\text{alt}}, \chi_3) = 0.$$

Therefore $\chi_{\text{alt}} = \chi_4 + \chi_5$ is the sum of two irreducible characters. First we find the dimensions of ρ_4 and ρ_5 using

$$1^2 + 4^2 + 5^2 + n_4^2 + n_5^2 = 60.$$

We obtain $n_4 = n_5 = 3$. The equations

$$(\chi_4, \chi_1 + \chi_2) = 0,\ (\chi_4, \chi_3) = 0$$

imply

$$\chi_4\left((123)\right) = 0,\ \chi_4\left((12)\,(34)\right) = -1.$$

The same argument shows

$$\chi_5\left((123)\right) = 0,\ \chi_5\left((12)\,(34)\right) = -1.$$

Finally denote

$$x = \chi_4\left((12345)\right),\ y = \chi_4\left((12354)\right)$$

and write down the equation $(\chi_4, \chi_4) = 1$. It is

$$\frac{1}{60}\left(9 + 15 + 12x^2 + 12y^2\right) = 1,$$

or

(1.2) $$x^2 + y^2 = 3.$$

On the other hand, $(\chi_4, \chi_1) = 0$, that gives

$$3 - 15 + 12\,(x + y) = 0,$$

or

(1.3) $$x + y = 1.$$

One can solve (1.2) and (1.3)

$$x = \frac{1+\sqrt{5}}{2}, y = \frac{1-\sqrt{5}}{2}.$$

The second solution

$$x = \frac{1-\sqrt{5}}{2}, y = \frac{1+\sqrt{5}}{2}$$

will give the last character χ_5.

2. Modules

Let R be a ring, usually we assume that $1 \in R$. An abelian group M is called a *(left) R-module* if there is a map $\alpha\colon R \times M \to M$, (we write $\alpha(a, m) = am$) satisfying

(1) $(ab)\,m = a\,(bm)$;
(2) $1m = m$;
(3) $(a + b)\,m = am + bm$;
(4) $a\,(m + n) = am + an$.

Example 1. If R is a field then any R-module M is a vector space over R.

Example 2. If $R = k(G)$ is a group algebra, then every R-module defines the representation $\rho\colon \to \mathrm{GL}(V)$ by the formula

$$\rho_s v = sv$$

for any $s \in G \subset k(G)$, $v \in V$. Conversely, every representation $\rho\colon G \to \mathrm{GL}(V)$ in a vector space V over k defines on V a $k(G)$-module structure by

$$\left(\sum_{s\in G} a_s s\right) v = \sum_{s\in G} a_s \rho_s v.$$

Thus, representations of G over k are $k(G)$-modules.

A *submodule* is a subgroup invariant under R-action. If $N \subset M$ is a submodule then the quotient M/N has the natural R-module structure. A module M is *simple* or (*irreducible*) if any submodule is either zero or M. A sum and an intersection of submodules is a submodule.

Example 3. If R is an arbitrary ring, then R is a left module over itself, where the action is given by the left multiplication. Submodules are left ideals.

For any two R-modules M and N one can define an abelian group $\mathrm{Hom}_R(M, N)$ and a ring $\mathrm{End}_R(M)$ in the manner similar to the group case. Schur's Lemma holds for simple modules in the following form.

Lemma 2.1. *Let M and N be simple modules and $\varphi \in \mathrm{Hom}_R(M, N)$, then either φ is an isomorphism or $\varphi = 0$. For a simple module M, $\mathrm{End}_R(M)$ is a division ring.*

Recall that for every ring R one defines R^{op} as the same abelian group with new multiplication $*$ given by

$$a * b = ba.$$

Lemma 2.2. *The ring* $\mathrm{End}_R(R)$ *is isomorphic to* R^{op}.

Proof. For each $a \in R$ define $\varphi_a \in \mathrm{End}(R)$ by the formula

$$\varphi_a(x) = xa.$$

It is easy to check that $\varphi_a \in \mathrm{End}_R(R)$ and $\varphi_{ba} = \varphi_a \circ \varphi_b$. Hence we constructed a homomorphism $\varphi : R^{\mathrm{op}} \to \mathrm{End}_R(R)$. To prove that φ is injective let $\varphi_a = \varphi_b$. Then $\varphi_a(1) = \varphi_b(1)$, i.e. $a = b$. To prove surjectivity of φ, note that for any $\gamma \in \mathrm{End}_R(R)$ one has

$$\gamma(x) = \gamma(x1) = x\gamma(1).$$

Therefore $\gamma = \varphi_{\gamma(1)}$. □

Lemma 2.3. *Let* $\rho_i : G \to \mathrm{GL}(V_i)$, $i = 1, \dots, l$ *be pairwise non-isomorphic irreducible representations of a group* G *over algebraically closed field* k, *and*

$$V = V_1^{\oplus m_1} \oplus \cdots \oplus V_l^{\oplus m_l}.$$

Then

$$\mathrm{End}_G(V) \cong \mathrm{End}_k(k^{m_1}) \times \cdots \times \mathrm{End}_k(k^{m_l}).$$

Proof. First, note that the Schur's Lemma implies that $\varphi\left(V_i^{\oplus m_i}\right) \subset V_i^{\oplus m_i}$ for any $\varphi \in \mathrm{End}_G(V)$, $i = 1, \dots, l$. Hence

$$\mathrm{End}_G(V) \cong \mathrm{End}_G\left(V_1^{\oplus m_1}\right) \times \cdots \times \mathrm{End}_G\left(V_l^{\oplus m_l}\right).$$

Therefore it suffices to prove the following

Lemma 2.4. *For an irreducible representation of* G *in* W

$$\mathrm{End}_G\left(W^{\oplus m}\right) \cong \mathrm{End}_k(k^m).$$

Proof. Let $p_i : W^{\oplus m} \to W$ denotes the projection onto the i-th component and $q_j : W \to W^{\oplus m}$ be the embedding of the j-th component. Let $\varphi \in \mathrm{End}_G(W^{\oplus m})$. Then by Schur's Lemma $p_i \circ \varphi \circ q_j = \varphi_{ij}\,\mathrm{Id}$ for some $\varphi_{ij} \in k$. Therefore we have the map $\Phi : \mathrm{End}_G(W^{\oplus m}) \to \mathrm{End}_k(k^m)$, (the latter is just the matrix ring) defined by $\Phi(\varphi) = (\varphi_{ij})$. Check yourself that Φ is an isomorphism. □

□

Theorem 2.5. *Let* k *be algebraically closed,* $\mathrm{char}\, k = 0$. *Then*

$$k(G) \cong \mathrm{End}_k(k^{n_1}) \times \cdots \times \mathrm{End}_k(k^{n_l}),$$

where $n_1, \dots, n_l$ *are dimensions of irreducible representations.*

Proof. By Lemma 2.2

$$\operatorname{End}_{k(G)}(k(G)) \cong k(G)^{\mathrm{op}} \cong k(G),$$

since $k(G)^{\mathrm{op}} \cong k(G)$ via $g \to g^{-1}$. On the other hand

$$k(G) = V_1^{\oplus n_1} \oplus \cdots \oplus V_l^{\oplus n_l},$$

since every irreducible $\rho_i : G \to \mathrm{GL}(V_i)$ appears with the multiplicity $n_i = \dim V_i$. Therefore Lemma 2.3 implies theorem. □

3. Finitely generated modules and Noetherian rings.

A module M is *finitely generated* if there exist $x_1, \dots, x_n \in M$ such that $M = Rx_1 + \cdots + Rx_n$.

Lemma 3.1. *Let*

$$0 \to N \xrightarrow{q} M \xrightarrow{p} L \to 0$$

be an exact sequence of R-modules. If M is finitely generated, then L is finitely generated. If M and L are finitely generated, then M is finitely generated.

Proof. First assertion is obvious. For the second let

$$L = Rx_1 + \dots Rx_n,\ N = Ry_1 + \dots Ry_m,$$

then $M = Rp^{-1}(x_1) + \cdots + Rp^{-1}(x_n) + Rq(y_1) + \cdots + Rq(y_m)$. □

Lemma 3.2. *The following conditions on a ring R are equivalent*

(1) *Every increasing chain of left ideals is finite, in other words for any sequence $I_1 \subset I_2 \subset \dots$, $I_n = I_{n+1} = I_{n+2} = \dots$ starting from some n;*
(2) *Every left ideal is finitely generated R-module.*

Proof. (1)⇒ (2). Assume that some left ideal I is not finitely generated. Then there exists an infinite sequence of $x_n \in I$ such that

$$x_{n+1} \notin Rx_1 + \dots Rx_n.$$

But then $I_n = Rx_1 + \dots Rx_n$ form an infinite increasing chain of ideals.

(2)⇒ (1). Let $I_1 \subset I_2 \subset \dots$ be an increasing chain of ideals . Let $I = \cup_n I_n$. Then $I = Rx_1 + \dots Rx_s$, where $x_j \in I_{n_j}$. Let m be maximal among $n_1, \dots, n_s$. Then $I = I_m$, and therefore the chain is finite. □

A ring satisfying the conditions of Lemma 3.2 is called *left Noetherian*.

Lemma 3.3. *Let R be a left Noetherian ring and M be a finitely generated R-module. Then every submodule of M is finitely generated.*

Proof. Let $M = Rx_1 + \dots Rx_n$, the there exists a surjective homomorphism $p : R \oplus \dots \oplus R \to M$, such that

$$p(r_1, \dots, r_n) = r_1 s_1 + \dots + r_n s_n.$$

As follows from the first part of Lemma 3.1, it suffices to prove the statement for a free module. It can be done by induction using the second part of Lemma 3.1.

□

Let R be a commutative ring. An element $x \in R$ is called *integral over* $\mathbb{Z}$ if $x^n + a_{n-1}x + \dots + a_0 = 0$ for some $a_i \in \mathbb{Z}$. This condition is equivalent to the condition that $\mathbb{Z}[x] \subset R$ is finitely generated $\mathbb{Z}$-module. Complex numbers integral over $\mathbb{C}$ are usually called algebraic integers. Obviously, if a rational number z is algebraic integer, then $z \in \mathbb{Z}$.

Lemma 3.4. *The set of integral elements in a commutative ring R is a subring.*

Proof. If $\mathbb{Z}[x]$ and $\mathbb{Z}[y]$ are finitely generated over $\mathbb{Z}$, then $\mathbb{Z}[x, y]$ is also finitely generated. Let $s \in \mathbb{Z}[x, y]$, then $\mathbb{Z}[s]$ is finitely generated since $\mathbb{Z}$ is Noetherian ring and we can apply Lemma 3.3.

□

4. The center of the group algebra $k(G)$

We have assumptions $\operatorname{char} k = 0$, $\bar{k} = k$, G is a finite group. Let $Z(G)$ denote the center of $k(G)$. It is obvious that

$$Z(G) = \left\{ \sum_{s \in G} f(s)\, s \mid f \in \mathcal{C}(G) \right\}.$$

On the other hand, by Theorem 2.5 we have

$$k(G) = \operatorname{End}_k(k^{n_1}) \times \dots \times \operatorname{End}_k(k^{n_l}).$$

Therefore $Z(G)$ is isomorphic to k^l as a commutative ring. Let e_i denote the identity element in $\operatorname{End}_k(k^{n_i})$. Then $e_1, \dots, e_l$ form a basis in $Z(G)$ and

$$e_i e_j = \delta_{ij} e_i,\ 1 = e_1 + \dots + e_l.$$

For an irreducible representation $\rho_j : G \to \mathrm{GL}(V_j)$ we have

$$\rho_j(e_i) = \delta_{ij} \operatorname{Id}. \tag{4.1}$$

Lemma 4.1. *If $\chi_i = \chi_{\rho_i}$, $n_i = \dim V_i$, then*

$$e_i = \frac{n_i}{|G|} \sum \chi_i\left(s^{-1}\right) s. \tag{4.2}$$

Proof. We need to check (4.1). Since $\rho_j(e_i) \in \operatorname{End}_G(V_j)$, we have $\rho_j(e_i) = \lambda Id$. To find λ calculate

$$\operatorname{tr} \rho_j(e_i) = \frac{n_i}{|G|} \sum \chi_i\left(s^{-1}\right) \chi_j(s) = \frac{n_i}{|G|} (\chi_i, \chi_j) = \delta_{ij} n_i.$$

□

Lemma 4.2. *Define $\omega_i : Z(G) \to k$ by the formula*

$$\omega_i\left(\sum a_s s\right) = \frac{1}{n_i} \sum a_s \chi_i(s).$$

Then ω_i is a homomorphism of rings and

$$\omega = (\omega_1, \dots, \omega_l) : Z(G) \to k^l$$

is an isomorphism.

Proof. Check that $\omega_i(e_j) = \delta_{ij}$ using again the orthogonality relation. □

Lemma 4.3. *Let $u = \sum a_s s \in Z(G)$. If all a_s are algebraic integers, then u is integral over $\mathbb{Z}$.*

Proof. Let $c \subset G$ be some conjugacy class and let

$$\delta_c = \sum_{s \in c} s.$$

If $c_1, \dots, c_l$ are disjoint conjugacy classes, then clearly $\mathbb{Z}\delta_{c_1} + \dots + \mathbb{Z}\delta_{c_l}$ is a subring in $Z(G)$. On the other hand, it is clearly a finitely generated $\mathbb{Z}$-module, and therefore every element of it is integral over $\mathbb{Z}$. But then for any set of algebraic integers $b_1, \dots, b_l$ the element $\sum b_i \delta_{c_i}$ is integral over $\mathbb{Z}$, which proves Lemma. □

Theorem 4.4. *The dimension of an irreducible representation divides $|G|$.*

Proof. For every $s \in G$, $\chi_i(s)$ is an algebraic integer. Therefore by Lemma 4.3 $u = \sum_{s \in G} \chi_i(s^{-1}) s$ is integral over $\mathbb{Z}$. Hence $\omega_i(u)$ is an algebraic integer. But

$$\omega_i(u) = \frac{1}{n_i} \sum \chi_i(s) \chi_i\left(s^{-1}\right) = \frac{|G|}{n_i} (\chi_i, \chi_i) = \frac{|G|}{n_i}.$$

Therefore $\frac{|G|}{n_i} \in \mathbb{Z}$. □

Theorem 4.5. *Let Z be the center of G. The dimension n of an irreducible representation divides $\frac{|G|}{|Z|}$.*

Proof. Consider

$$\rho_m = \rho^{\boxtimes m} : G \times \dots \times G \to \mathrm{GL}\left(V^{\otimes m}\right).$$

Then $\operatorname{Ker} \rho_m$ contains a subgroup

$$Z'_m = \{(z_1, \dots, z_m) \in Z^m \mid z_1 z_2 \dots z_m = 1\}.$$

If ρ is irreducible, then ρ_m is irreducible, and $\dim \rho_m = (\dim \rho)^m$ divides $|G^m/Z'_m| = \frac{|G|^m}{|Z|^{m-1}}$. Since this is true for any m, then $\dim \rho$ divides $\frac{|G|}{|Z|}$ (check yourself using prime factorization). $\square$