

REPRESENTATION THEORY

WEEK2

VERA SERGANOVA

1. CHARACTERS

For any finite-dimensional representation $\rho : G \rightarrow \text{GL}(V)$ its *character* is a function $\chi_\rho : G \rightarrow k$ defined by

$$\chi_\rho(s) = \text{tr } \rho_s.$$

It is easy to see that characters have the following properties

- (1) $\chi_\rho(1) = \dim \rho$;
- (2) if $\rho \cong \sigma$, then $\chi_\rho = \chi_\sigma$;
- (3) $\chi_{\rho \oplus \sigma} = \chi_\rho + \chi_\sigma$;
- (4) $\chi_{\rho \otimes \sigma} = \chi_\rho \chi_\sigma$;
- (5) $\chi_{\rho^*}(s) = \chi_\rho(s^{-1})$;
- (6) $\chi_\rho(sts^{-1}) = \chi_\rho(t)$.

Example 1. If R is a regular representation, then $\chi_R(s) = 0$ for any $s \neq 1$ and $\chi_R(1) = |G|$.

Example 2. Let $\rho : G \rightarrow \text{GL}(V)$ be a representation. Recall that $\rho \otimes \rho = \text{sym} \oplus \text{alt}$, where $\text{alt} : G \rightarrow \text{GL}(\Lambda^2 V)$ and $\text{sym} : G \rightarrow \text{GL}(S^2 V)$. Let us calculate χ_{sym} and χ_{alt} . For each $s \in G$ let $\lambda_1, \dots, \lambda_n$ be eigenvalues of ρ_s taken with multiplicities. Then the eigenvalues of alt_s are $\lambda_i \lambda_j$ for all $i < j$ and the eigenvalues of sym_s are $\lambda_i \lambda_j$ for $i \leq j$. Hence

$$\chi_{\text{sym}}(s) = \sum_{i \leq j} \lambda_i \lambda_j, \quad \chi_{\text{alt}}(s) = \sum_{i < j} \lambda_i \lambda_j,$$

and therefore

$$\chi_{\text{sym}}(s) - \chi_{\text{alt}}(s) = \sum_i \lambda_i^2 = \text{tr } \rho_{s^2} = \chi_\rho(s^2).$$

On the other hand by properties (3) and (4)

$$\chi_{\text{sym}}(s) + \chi_{\text{alt}}(s) = \chi_{\rho \otimes \rho}(s) = \chi_\rho^2(s).$$

Thus, we get

$$\chi_{\text{sym}}(s) = \frac{\chi_\rho^2(s) + \chi_\rho(s^2)}{2}, \quad \chi_{\text{alt}}(s) = \frac{\chi_\rho^2(s) - \chi_\rho(s^2)}{2}.$$

Starting from this point we assume that G is finite and k is algebraically closed of characteristic 0.

Introduce the non-degenerate symmetric bilinear form on the space of functions $\mathcal{F}(G)$ by the formula

$$(f, g) = \frac{1}{|G|} \sum_{s \in G} f(s^{-1}) g(s).$$

Theorem 1.1. *Let ρ, σ be irreducible. If ρ and σ are not isomorphic, then $(\chi_\rho, \chi_\sigma) = 0$. If ρ and σ are isomorphic, then $(\chi_\rho, \chi_\sigma) = 1$.*

Proof. Let V be the space of the representation ρ and W be the space of σ . Denote $n = \dim V$, $m = \dim W$. Choose a basis v_1, \dots, v_n in V , w_1, \dots, w_m in W . Define $P(i, j) : W \rightarrow W$ by

$$P(i, j) v_k = \delta_{jk} w_i.$$

Lemma 1.2. *For any $T \in \text{Hom}_k(V, W)$ let*

$$\bar{T} = \frac{1}{|G|} \sum_{s \in G} \sigma_s \circ T \circ \rho_{s^{-1}}.$$

Then $\bar{T} \in \text{Hom}_G(V, W)$. If $V = W$, then $\text{tr } T = \text{tr } \bar{T}$.

Proof. Direct calculations. □

For any $T \in \text{Hom}(V, W)$ let T_{kl} denote the corresponding matrix entry. For example, $P(i, j)_{kl} = \delta_{ik} \delta_{jl}$. Then

$$\bar{P}(i, j)_{kl} = \frac{1}{|G|} \sum_{s \in G} (\sigma_s)_{ki} (\rho_{s^{-1}})_{jl}.$$

If σ and ρ are not isomorphic, then by Schur's Lemma

$$\bar{P}(i, j)_{kl} = 0$$

for all i, j, k, l . In particular, $\bar{P}(i, j)_{ij} = 0$ and therefore

$$\sum_{i=1}^m \sum_{j=1}^n \bar{P}(i, j)_{ij} = \frac{1}{|G|} \sum_{i=1}^m \sum_{j=1}^n \sum_{s \in G} (\sigma_s)_{ii} (\rho_{s^{-1}})_{jj} = 0.$$

But

$$\sum_{i=1}^m \sum_{j=1}^n (\sigma_s)_{ii} (\rho_{s^{-1}})_{jj} = \chi_\sigma(s) \chi_\rho(s^{-1}).$$

Hence

$$\frac{1}{|G|} \sum_{s \in G} \chi_\sigma(s) \chi_\rho(s^{-1}) = (\chi_\rho, \chi_\sigma) = 0.$$

Let now $\rho \cong \sigma$. Then by Property (2), we may assume $\rho = \sigma$. Then by Schur's Lemma

$$\bar{P}(i, j) = \lambda \text{Id}.$$

Since $\text{tr } \bar{P}(i, j) = \text{tr } P(i, j) = \delta_{ij}$, we have

$$\bar{P}(i, j) = \frac{\delta_{ij}}{n} \text{Id}.$$

Then

$$\sum_{i=1}^n \sum_{j=1}^n \bar{P}(i, j)_{ij} = \sum_{i=1}^n \sum_{j=1}^n \frac{\delta_{ij}}{n} = 1,$$

which implies $(\chi_\rho, \chi_\rho) = 1$. □

Corollary 1.3. *Let $\rho = m_1\rho_1 \oplus \dots \oplus m_l\rho_l$ be decomposition into the sum of irreducibles. Then $m_i = (\chi_\rho, \chi_{\rho_i})$.*

The number m_i is called the *multiplicity* of an irreducible representation ρ_i in ρ .

Corollary 1.4. *A representation ρ is irreducible iff $(\chi_\rho, \chi_\rho) = 1$.*

Corollary 1.5. *Every irreducible representation ρ appears in the regular representation with multiplicity $\dim \rho$.*

Proof.

$$(\chi_\rho, \chi_R) = \frac{1}{|G|} \chi_\rho(1) \chi_R(1) = \dim \rho.$$

□

Corollary 1.6. *Let ρ_1, \dots, ρ_l be all (up to isomorphism) irreducible representations of G and $n_i = \dim \rho_i$. Then*

$$n_1^2 + \dots + n_l^2 = |G|.$$

Example 3. Let G act on a finite set X and

$$k(X) = \left\{ \sum_{x \in X} b_x x \mid b_x \in k \right\}.$$

Define $\rho : G \rightarrow \text{GL}(k(X))$ by

$$\rho_s \sum_{x \in X} b_x x = \sum_{x \in X} b_x s(x).$$

It is easy to check that ρ is a representation and

$$\chi_\rho(s) = |\{x \in X \mid s(x) = x\}|.$$

Clearly, ρ contains a trivial subrepresentation. To find the multiplicity of a trivial representation in ρ we have to calculate $(1, \chi_\rho)$.

$$(1, \chi_\rho) = \frac{1}{|G|} \sum_{s \in G} \chi_\rho(s) = \frac{1}{|G|} \sum_{s \in G} \sum_{s(x)=x} 1 = \frac{1}{|G|} \sum_{x \in X} \sum_{s \in G_x} 1 = \frac{1}{|G|} \sum_{x \in X} |G_x|,$$

where

$$G_x = \{s \in G \mid s(x) = x\}.$$

Let $X = X_1 \cup \dots \cup X_m$ be the disjoint union of orbits. Then $|G_x| = \frac{|G|}{|X_i|}$ for each $x \in X_i$ and therefore

$$(1, \chi_\rho) = \frac{1}{|G|} \sum_{i=1}^m \sum_{x \in X_i} \frac{|G|}{|X_i|} = m.$$

Now let us evaluate (χ_ρ, χ_ρ) .

$$(\chi_\rho, \chi_\rho) = \frac{1}{|G|} \sum_{s \in G} \left(\sum_{s(x)=x} 1 \right)^2 = \frac{1}{|G|} \sum_{s \in G} \sum_{s(x)=x, s(y)=y} 1 = \frac{1}{|G|} \sum_{(x,y) \in X \times X} |G_x \cap G_y|.$$

Let σ be the representation associated with the action of G on $X \times X$. Then the last formula implies

$$(\chi_\rho, \chi_\rho) = (1, \chi_\sigma).$$

Thus, ρ is irreducible iff $|X| = 1$, and ρ has two irreducible components iff the action of G on $X \times X$ with removed diagonal is transitive or $|X| = 2$.

Let

$$\mathcal{C}(G) = \{f \in \mathcal{F}(G) \mid f(sts^{-1}) = f(t)\}.$$

It is easy to check that the restriction of (\cdot, \cdot) on $\mathcal{C}(G)$ is non-degenerate.

Theorem 1.7. *The characters of irreducible representations of G form an orthonormal basis in $\mathcal{C}(G)$.*

Proof. We have to show that if $f \in \mathcal{C}(G)$ and $(f, \chi_\rho) = 0$ for any irreducible ρ , then $f = 0$. The following lemma is straightforward.

Lemma 1.8. *Let $\rho : G \rightarrow \text{GL}(V)$ be a representation, $f \in \mathcal{C}(G)$ and*

$$T = \frac{1}{|G|} \sum_{s \in G} f(s^{-1}) \rho_s.$$

Then $T \in \text{End}_G V$ and $\text{tr } T = (f, \chi_\rho)$.

Thus, for any irreducible ρ we have

$$(1.1) \quad \frac{1}{|G|} \sum_{s \in G} f(s^{-1}) \rho_s = 0.$$

But then the same is true for any representation ρ , since any representation is a direct sum of irreducibles. Apply (?_{equ1}?) for the case when $\rho = R$ is the regular representation. Then

$$\frac{1}{|G|} \sum_{s \in G} f(s^{-1}) R_s 1 = \frac{1}{|G|} \sum_{s \in G} f(s^{-1}) s = 0.$$

Hence $f(s^{-1}) = 0$ for all $s \in G$, i.e. $f = 0$ □

Corollary 1.9. *The number of isomorphism classes of irreducible representations equal the number of the conjugacy classes.*