REPRESENTATION THEORY.
WEEK 14

1. Applications of quivers

Two rings A and B are *Morita equivalent* if the categories of $A-$ modules and B-modules are equivalent. A projective finitely generated A-module P is a *projective generator* if any other projective finitely generated A-module is isomorphic to a direct summand of $P^{\oplus n}$ for some n.

Theorem 1.1. *A and B are Morita equivalent iff there exists a projective generator P in $A-\mod$ such that $B \cong \operatorname{End}_A(P)$. The functor $X \mapsto \operatorname{Hom}_A(P, X)$ establishes the equivalence between $A-\mod$ and $B-\mod$.*

For the proof see, for example, Bass "Algebraic K-theory".

Assume now that C is a finite-dimensional algebra over algebraically closed field k. Let $P_1, \dots, P_n$ be a set of representatives of isomorphism classes of indecomposable projective C-modules. Then $P = P_1 \oplus \cdots \oplus P_n$ is a projective generator, and $A = \operatorname{End}_C(P)$ is Morita equivalent to C.

Example 1.2. Let C be semisimple, then $C \cong \operatorname{Mat}_{m_1}(k) \times \cdots \times \operatorname{Mat}_{m_n}(k)$, and $A \cong k^n$. Let

$$C = \left\{ \binom{XY}{0Z} \in \operatorname{Mat}_{p+q}(k) \mid X \in \operatorname{Mat}_p(k), Y \in \operatorname{Mat}_{p,q}(k), Z \in \operatorname{Mat}_q(k) \right\}.$$

Then

$$A = \left\{ \binom{xy}{0z} \mid x, y, z \in k \right\}.$$

Let R be the radical of C. Then each indecomposable projective P_i has the filtration $P_i \supset RP_i \supset R^2P_i \supset \cdots \supset 0$ such that $R^jP_i/R^{j+1}P_i$ is semisimple for all j. Recall that P_i/RP_i is simple (lecture notes 9), hence $\operatorname{Hom}_C(P_i, P_j/RP_j) = 0$ if $i \neq j$. Define the quiver Q in the following way. Vertices are enumerated by indecomposable projective modules $P_1, \dots, P_n$, the number of arrows $i \to j$ equals $\dim \operatorname{Hom}_C(P_i, RP_j/R^2P_j)$. We construct a surjective homomorphism $\phi\colon k(Q) \to A$. (This construction is not canonical). Fist set $\phi(e_i) = \operatorname{Id}_{P_i}$. Let $\gamma_1, \dots, \gamma_s$ be the set of arrows from i to j, choose a basis $\eta_1, \dots, \eta_s \in \operatorname{Hom}_C(P_i, RP_j/R^2P_j)$, each η_l can be lifted to $\xi_l \in \operatorname{Hom}_C(P_i, RP_j)$ as P_i is projective. Define $\phi(\gamma_l) = \xi_l$. Now ϕ extends in the unique way to the whole $k(Q)$ since $k(Q)$ is generated by idempotents e_i and arrows.

Since ϕ is surjective, then $A \cong k(Q)/I$ for some two-sided ideal $I \subset k(Q)$. The pair Q and an ideal I in $k(Q)$ is called a *quiver with relations.* The problem of classification of indecomposable C-modules is equivalent to the problem of classification

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of indecomposable representations of Q satisfying relations I. In some cases such quiver approach is very useful.

Example 1.3. Let k be the algebraic closure of $\mathbb{F}_3$ and $C = k[S_3]$. In lecture notes 9 we showed that C has two indecomposable projectives $P_+ = \mathrm{Ind}_{S_2}^{S_3}$ triv and $P_- = \mathrm{Ind}_{S_2}^{S_3}$ sgn. The quiver Q is

$$\bullet \iff {}^{\alpha}_{\beta}\bullet$$

with relations $\alpha\beta\alpha = 0$, $\beta\alpha\beta = 0$. The quiver itself is $\hat{\mathrm{A}}_2$, indecomposable representations have dimensions (m,m), $(m+1,m)$ and $(m,m+1)$. Since we have the precise description, it is not difficult to see that only six indecomposable representations satisfy the relations. They are

$$k \iff 0;\ 0 \iff k;\ k \iff k,\ \alpha = 1, \beta = 0 \text{ or } \alpha = 0, \beta = 1,$$

$$k^2 \iff k,\ \alpha = (10), \beta = \binom{0}{1};\ k \iff k^2,\ \alpha = \binom{0}{1}, \beta = (10).$$

The first two representations correspond to irreducible representations triv and sgn, the last two are projectives. Two representations of dimension (1,1) correspond to the quotients of P_+ and P_- by the minimal submodules.

In fact one can apply the quiver approach to any category $\mathcal{C}$ which satisfies the following conditions

(1) All objects have finite length;
(2) Any object has a projective resolution;
(3) For any two objects X, Y, $\mathrm{Hom}(X,Y)$ is a vector space over an algebraically closed field k.

We do not need the assumption that the number of simple or projective objects is finite. We illustrate this in the following example.

Example 1.4. Let Λ be the Grassmann algebra with two generators, i.e. $\Lambda = k < x, y > / (x^2, y^2, xy + yx)$. Consider the $\mathbb{Z}$-grading $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2$, where $\Lambda_0 = k$, Λ_1 is the span of x and y, $\Lambda_2 = kxy$. Let $\mathcal{C}$ denote the category of graded Λ-modules. In other words, objects are Λ-modules $M = \bigoplus_{i\in\mathbb{Z}} M_i$, such that $\Lambda_i M_j \subset M_{i+j}$ and morphisms preserve the grading. All projective modules are free. An indecomposable projective module P_i is isomorphic to Λ with shifted grading $\deg(1) = i$. Thus, the quiver Q has infinitely many vertices enumerated by $\mathbb{Z}$:

$$\cdots \Leftarrow^{\alpha_i}_{\beta_i} \bullet \Leftarrow^{\alpha_{i+1}}_{\beta_{i+1}} \bullet \Leftarrow^{\alpha_{i+2}}_{\beta_{i+2}} \bullet \Leftarrow^{\alpha_{i+3}}_{\beta_{i+3}} \cdots$$

Here $\alpha_{i+1}, \beta_{i+1} \in \mathrm{Hom}(P_{i+1}, P_i)$, $\alpha_{i+1}(1) = x$, $\beta_{i+1}(1) = y$. Relations are $\alpha_i\alpha_{i+1} = \beta_i\beta_{i+1} = 0$, $\alpha_i\beta_{i+1} + \beta_i\alpha_{i+1} = 0$.

Let us classify the indecomposable representations of above quiver. Assume first that, that there exists $v \in X_{i+1}$ such that $\alpha_i\beta_{i+1} \neq 0$, Then the subrepresentation V spanned by v, $\alpha_{i+1}v$, $\beta_{i+1}v$, $\alpha_i\beta_{i+1}v$ splits as a direct summand in X. If X is indecomposable, then $X = V$. The corresponding object in $\mathcal{C}$ is P_{i+1}.

Now assume that $\alpha_i\beta_{i+1}X_{i+1} = 0$ for any $i \in \mathbb{Z}$. That is equivalent to putting the new relations for Q: every path of length 2 is zero. Consider the subspaces

$$W_i = \operatorname{Im}\alpha_{i+1} + \operatorname{Im}\beta_{i+1} \subset X_i,\ Z_{i+1} = \operatorname{Ker}\alpha_{i+1} \cap \operatorname{Ker}\beta_{i+1} \subset X_{i+1}.$$

One can find $U_i \subset X_i$ and $Y_{i+1} \subset X_{i+1}$ such that $X_i = U_i \oplus W_i$, $X_{i+1} = Z_{i+1} \oplus Y_{i+1}$. Check that $W_i \oplus Y_{i+1}$ is a subrepresentation, which splits as a direct summand in X. If X is indecomposable and $W_i \neq 0$, then $X = W_i \oplus Y_{i+1}$. Thus, we reduced our problem to Kronecker quiver $\bullet \Leftarrow \bullet$! There is the obvious bijection between indecomposable non-projective objects from $\mathcal{C}$ and the pairs (Y, i), where Y is an indecomposable representation of Kronecker quiver, $i \in \mathbb{Z}$ (defines the grading).

Remark 1.5. The last example is related to the algebraic geometry as the derived category of $\mathcal{C}$ is equivalent to the derived category of coherent sheaves on $\mathbb{P}^1$.

Remark 1.6. If in the last example we increase the number of generators in Λ, then the problem becomes wild (definition below).

Let C be a finite-dimensional algebra. We say that C is *finitely represented* if C has finitely many indecomposable representations. We call C *tame* if for each $d \subset \mathbb{Z}_{>0}$, there exist a finite set $M_1, \ldots, M_r$ of $C - k\,[x]$ bimodules (free of rank d over $k\,[x]$) such that every indecomposable representation of C of dimension d is isomorphic to $M_i \otimes_{k[x]} k\,[x] \,/\, (x - \lambda)$ for some $i \leq r$, $\lambda \in k$. Finally, C is *wild* if there exists a $C - k < x, y >$ bimodule M such that the functor $X \mapsto M \otimes_{k<x,y>} X$ preserves indecomposability and is faithful. We formulate here without proof the following results.

Theorem 1.7. *Every finite-dimensional algebra over algebraically closed field k is either finitely represented or tame or wild*

Theorem 1.8. *Let Q be a connected quiver without oriented cycles. Then $k\,(Q)$ is finitely represented iff Q is Dynkin, $k\,(Q)$ is tame iff Q is affine.*

Theorem 1.9. *Let Alg_n be the algebraic variety of all n-dimensional algebras over k. Then the set of finitely represented algebras is Zariski open in Alg_n.*

2. Frobenius algebras

Let A be a finite-dimensional algebra over k. Recall that we denote by D the functor $\mod - A \to A - \mod$, such that $D\,(X) = X^*$. Recall also that D maps projective modules to injective and vice versa.

A finite-dimensional A algebra over k is called a *Frobenius algebra* if $D\,(A_A)$ is isomorphic to A, where A_A is the right A-module over itself.

Theorem 2.1. *The following conditions on A are equivalent*

1. *A is a Frobenius algebra;*
2. *There exists a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on A such that $\langle ab, c \rangle = \langle a, bc \rangle$;*

(3) *There exists* $\lambda \in A^*$ *such that* $\operatorname{Ker}\lambda$ *does not contain non-trivial left or right ideals.*

Proof. A form $\langle \cdot, \cdot \rangle$ gives an isomorphism $\mu \colon A \to A^*$ by the formula $x \to \langle \cdot, x \rangle$. The condition $\langle ab, c \rangle = \langle a, bc \rangle$ is equivalent to μ being a homomorphism of modules. A linear functional λ can be constructed by $\lambda(x) = \langle 1, x \rangle$. Conversely, given λ, one can define $\langle x, y \rangle = \lambda(xy)$. The condition $\operatorname{Ker}\lambda$ does not contains non-trivial one-sided ideals is equivalent to the condition that the left and right kernels of $\langle \cdot, \cdot \rangle$ are zero. □

Lemma 2.2. *Let A be a Frobenius algebra. An A-module X is projective iff it is injective.*

Proof. A projective module X is a direct summand of a free module, but a free module is injective as $D(A_A)$ is isomorphic to A. Hence, X is injective. By duality an injective module is projective. □

Example 2.3. A group algebra $k(G)$ is Frobenius. Take

$$\lambda\left(\sum_{g\in G} a_g g\right) = a_1.$$

The corresponding bilinear form is symmetric.

A Grassmann algebra $\Lambda = k < x_1, \dots, x_n > / (x_i x_j + x_j x_i)$ is Frobenius. Put

$$\lambda\left(\sum_{i_1<\dots<i_k} c_{i_1\dots i_k} x_{i_1}\dots x_{i_k}\right) = c_{12\dots n}.$$

In a sense Frobenius algebras generalize group algebras. For example, if $T \in \operatorname{Hom}_k(X, Y)$ for two $k(G)$-modules X and Y then

$$\bar{T} = \sum_{g\in G} gTg^{-1} \in \operatorname{Hom}_G(X, Y).$$

This idea of taking average over the group is very important in representation theory. It has an analog for Frobenius algebras.

Choose a basis $e_1, \dots, e_n$ in a Frobenius algebra A. Let $f_1, \dots, f_n$ be the dual basis, i.e.

$$\langle f_i, e_j \rangle = \delta_{ij}. \tag{2.1}$$

Every $a \in A$ can be written

$$a = \sum \langle f_i, a \rangle e_i = \sum \langle a, e_i \rangle f_i. \tag{2.2}$$

and

$$\sum a e_i \otimes f_i = \sum \langle f_j, a e_i \rangle e_j \otimes f_i = \sum \langle f_j a, e_i \rangle e_j \otimes f_i = \sum e_j \otimes f_j a. \tag{2.3}$$

Lemma 2.4. *Let X and Y be A-modules, $T \in \operatorname{Hom}_k(X, Y)$. Then $\bar{T} = \sum e_i T f_i \in \operatorname{Hom}_A(X, Y)$.*

Proof. Direct calculation using (2.2) and (2.3). □

Example 2.5. If $A = k(G)$, the dual bases can be chosen as $\{g\}_{g\in G}$ and $\{g^{-1}\}_{g\in G}$. Hence $\bar{T} = \sum gTg^{-1}$.

In Frobenius algebra one can use the following criterion of projectivity.

Theorem 2.6. *An A-module X is injective (hence projective) if there exists $T \in \operatorname{End}_k(X)$ such that $\bar{T} = \operatorname{Id}$.*

Proof. First, assume the existence of T. We have to show that X is injective, in other words, for any embedding $\varepsilon\colon X \to Y$ there exists $\pi \in \operatorname{Hom}_A(Y, X)$ such that $\pi \circ \varepsilon = Id$. There exists $p \in \operatorname{Hom}_k(Y, X)$ such that $p \circ \varepsilon = Id$. Put $\pi = \sum e_i T p f_i$. Then for any $x \in X$ we have

$$\pi(\varepsilon(x)) = \sum e_i T p f_i(\varepsilon(x)) = \sum e_i T(p\varepsilon(f_i x)) = \sum e_i T(f_i x) = \bar{T}x = \operatorname{Id}.$$

Here we use $f_i\varepsilon = \varepsilon f_i$. By Lemma 2.4 $\pi \in \operatorname{Hom}_A(X, Y)$.

Now assume that X is injective. Define the map $\delta\colon X \to A \otimes_k X$ by the formula

$$f(x) = \sum e_i \otimes f_i x.$$

Then $f \in \operatorname{Hom}_A(X, A \otimes_k X)$ by (2.3). It is obvious that f is injective. Thus, we may consider X as a submodule of X, moreover X is a direct summand because X is injective. So we have a projector $\tau : A \otimes_k X \to X$. Let $S \in \operatorname{Hom}_k(A \otimes_k X, A \otimes_k X)$ be defined by the formula

$$S(a \otimes x) = \langle 1, a\rangle 1 \otimes x.$$

Then

$$\bar{S}(a \otimes x) = \sum e_i S(f_i a \otimes x) = \sum \langle 1, f_i a\rangle e_i \otimes x = \sum \langle f_i, a\rangle e_i \otimes x = a \otimes x$$

due to (2.2). Put $T = \tau \circ S \circ \delta$. Then $\bar{T} = \operatorname{Id}$. □

3. Relative projective and injective modules in group algebra

Let H be a subgroup of a group G. A $k(G)$-module X is *H-injective* if any exact sequence of $k(G)$-modules

$$0 \to X \to Y \to Z \to 0,$$

which splits over $k(H)$, splits over $k(G)$.

In the similar way one defines *H-projective* module.

Let $\{g_1, \dots, g_r\}$ be a set of representatives in the set of left cosets G/H. For any $k(G)$-modules X, Y, and $T \in \operatorname{Hom}_H(X, Y)$ put

$$\bar{T} = \sum g_i T g_i^{-1}.$$

Prove yourself the following

Lemma 3.1. *$\bar{T}$ does not depend on a choice of representatives and $\bar{T} \in \mathrm{Hom}_G(X, Y)$.*

Theorem 3.2. *The following conditions on $k(G)$-module X are equivalent*

1. *X is H-injective;*
2. *X is a direct summand in $\mathrm{Ind}_H^G X$;*
3. *X is H-projective;*
4. *There exists $T \in \mathrm{End}_H(X)$ such that $\bar{T} = \mathrm{Id}$.*

Proof. This theorem is very similar to Theorem 2.6. To prove $1 \Rightarrow 2$ check that $\delta \colon X \to \mathrm{Ind}_H^G X$ defined by the formula

$$\delta(x) = \sum g_i \otimes g_i^{-1} x,$$

defines an embedding of X. By injectivity X is a direct summand of $\mathrm{Ind}_H^G X$.

To prove $3 \Rightarrow 2$ use the projection $\mathrm{Ind}_H^G X \to X$ defined by $g \otimes x \mapsto gx$.

Now prove $2 \Rightarrow 4$. Define $S\colon \mathrm{Ind}_H^G X \to \mathrm{Ind}_H^G X$ by

$$S\left(\sum g_i \otimes x_i\right) = 1 \otimes x_1,$$

here we assume that $g_1 = 1$. Check that $S \in \mathrm{End}_H\left(\mathrm{Ind}_H^G X\right)$ and $\bar{S} = Id$. Then obtain $T = \tau \circ S \circ \delta$, where $\tau\colon \mathrm{Ind}_H^G X \to X$ be the projection such that $\tau \circ \delta = \mathrm{Id}$.

Prove yourself $4 \Rightarrow 1$ and $4 \Rightarrow 3$ similarly to the first part of the proof of Theorem 2.6. □

The following corollary is important for us. Let p be prime. Recall that if $|G| = p^s r$ with $(p, r) = 1$, then there exists a subgroup P of order p^s. It is called a Sylow subgroup. Two Sylow p-subgroups are conjugate in G.

Corollary 3.3. *Let $\mathrm{char}\, k = p$ and P be a Sylow p-subgroup. Then every $k(G)$-module X is P-injective.*

Proof. We have to check condition (4) from Theorem 3.2. But $r = [G : P]$ is invertible in k. So we can put $T = \frac{1}{r}\mathrm{Id}$. □

4. Finitely represented group algebras

Let $\mathrm{char}\, k = p$, $|G| = p^s r$ with $(p, r) = 1$.

Lemma 4.1. *Let H be a cyclic p-group, i.e. $|H| = p^s$. Then there are exactly p^s isomorphism classes of indecomposable representations of H over k, exactly one for each dimension. More precisely each indecomposable L_m of dimension $m \leq p^s$ is isomorphic to $k(H)/(g-1)^m$, where g is a generator of H.*

Proof. Since $k(H) \cong k[\alpha]/\alpha^{p^s}$, where $\alpha = g - 1$, the corresponding quiver is the loop quiver with one relation $\alpha^{p^s} = 0$. Hence α is a nilpotent Jordan block of order $\leq p^s$. □

Theorem 4.2. *If a Sylow p-subgroup of G is cyclic, then $k(G)$ is finitely represented. Moreover, the number of indecomposable $k(G)$-modules is not greater than $|G|$.*

Proof. By Corollary 3.3 every indecomposable $k(G)$-module is P-injective. Therefore, any indecomposable X is a direct summand in $\mathrm{Ind}_P^G L_i$ for some i. Clearly, the number of such direct summands is finite. Now we will obtain the upper bound on the number of indecomposable representations. Let X be an indecomposbale $k(G)$-module, then by injectivity of X, X is a direct summand in $\mathrm{Ind}_P^G X$. Decompose X into a direct sum of indecomposable $k(P)$-modules, then X must be a direct summand in $\mathrm{Ind}_P^G L_i$ for some P-indecomposable summand L_i of X. Hence $\dim X \geq \dim L_i = i$. So if $\dim X = i$, then X can be realized as a summand in $\mathrm{Ind}_P^G(L_j)$ for some $j \leq i$. To calculate the total number of non-isomorphic indecomposable $k(G)$-modules, we can count in each $\mathrm{Ind}_P^G L_i$ only indecomposable $k(G)$-components of dimension $\geq i$ since others are realized in $\mathrm{Ind}_P^G L_j$ for $j < i$. Since there is no more than r such components for each i, the total number of non-isomorphic indecomposable $k(G)$-modules is not greater than $p^s r = |G|$. □

Lemma 4.3. *If P is a non-cyclic p-group, then P contains a normal subgroup N such that $P/N \cong \mathbb{Z}_p \times \mathbb{Z}_p$.*

Proof. If P is abelian, the statement follows from the classification of finite abelian groups. If P is not abelian, then P has a non-trivial center Z, and P/Z is not cyclic. The statement follows by induction on $|P|$. □

Lemma 4.4. *The group $S = \mathbb{Z}_p \times \mathbb{Z}_p$ has an indecomposable representation of dimension n for each $n \in \mathbb{Z}_{\geq 0}$.*

Proof. Let g and h be two generators of S, $\alpha = g - 1$, $\beta = h - 1$. Then $A = k(S)/(\alpha^2, \beta^2, \alpha\beta, \beta\alpha)$ is the subalgebra of $k(Q)$ for Kronecker quiver Q. In particular, one can see easily that every indecomposable representation of Q remains indecomposable after restriction to A. This implies the Lemma. □

Theorem 4.5. *If a p-Sylow subgroup of G is not cyclic, then G has an indecomposable representation of arbitrary high dimension.*

Proof. By Lemma 4.3 and Lemma 4.4, P has an indecomposable representation Y of dimension n for any positive integer n. Decompose $\mathrm{Ind}_P^G Y$ into direct sum of indecomposable $k(G)$-modules. At least one component X contains Y as an indecomposable $k(P)$ component. Hence $\dim X \geq n$. □

Corollary 4.6. *The group algebra $k(G)$ is finitely represented over a field of characteristic p iff a Sylow p-subgroup of G is cyclic.*