REPRESENTATION THEORY
WEEK 12

1. Reflection functors

Let Q be a quiver. We say a vertex $i \in Q_0$ is $+$-*admissible* if all arrows containing i have i as a target. If all arrows containing i have i as a source, we call i $-$-*admissible.* By $\sigma_i(Q)$ we denote the quiver obtained from Q by inverting all arrows containing i.

Let i be a $+$-admissible vertex and $Q' = \sigma_i(Q)$. Let us introduce the functor F_i^+: $\mathrm{Rep}_Q \to \mathrm{Rep}_{Q'}$. Let X be a representation of Q. Define $X' = F_i^+ X$ as follows. If $j \neq i$, then $X'_j = X_j$. Put $X'_i = \operatorname{Ker} h$, where

$$h = \sum_{\gamma=(j\to i)\in Q_1} \rho_\gamma : \bigoplus X_j \to X_i,$$

For each $\gamma = (i \to j) \in Q'$ define $\rho'_\gamma : X'_i \to X_j = X'_j$ as the natural projection on the component $X_j \in \bigoplus X_j$.

If i is a $-$-admissible vertex and $Q' = \sigma_i(Q)$ one can define the functor F_i^-: $\mathrm{Rep}_Q \to \mathrm{Rep}_{Q'}$ as follows. Let $X' = F_i^-(X)$, where $X'_j = X_j$ for $i \neq j$, and $X'_i = \operatorname{Coker} \widetilde{h}$, where

$$\widetilde{h} = \sum_{\gamma=(i\to j)\in Q_1} \rho_\gamma : X_i \to \oplus X_j,$$

and for each $\gamma = (j \to i) \in Q'$ define $\rho'_\gamma \colon X_j = X'_j \to X'_i$ by restriction of the projection $\oplus X_j \to \operatorname{Coker} \widetilde{h}$ to X_j.

Example. Let Q be the quiver $1 \to 2$, and X is the representation $k \to 0$, then $F_1^-(X) = 0$ and $F_2^+(X)$ is $k \leftarrow k$.

It is easy to check that F_i^+ is left-exact (maps an injection to an injection) and F_i^- is right exact (maps a surjection to a surjection). Let L_i denote the representation of Q which has k in the vertex i and zero in all other vertices. Then $F_i^+(L_i) = 0$ and $F_i^-(L_i) = 0$.

Theorem 1.1. *Let X be an indecomposable representation of Q and i be a $+$-admissible vertex. Then $F_i^+(X) = 0$ iff $X \cong L_i$. Otherwise $X' = F_i^+(X)$ is indecomposable,*

$$\dim X'_i = -\dim X_i + \sum_{(j\to i)} \dim X_j \qquad (1.1)$$

and $F_i^- F_i^+(X) \cong X$.

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If i is $-$-admissible vertex and X is indecomposable, then $F_i^-(X) = 0$ iff $X \cong L_i$. Otherwise $F_i^-(X)$ is indecomposable, the dimension of $F_i^-(X)$ can be calculated by the same formula (1.1) *and $F_i^+ F_i^-(X) \cong X$.*

Proof. Note that if $X \ncong L_i$, then h must be surjective because of indecomposability of X, hence the formula (1.1) holds. Furthermore, we have the following exact sequence

$$0 \to X_i' \xrightarrow{\widetilde{h}} \bigoplus_{(j\to i)\in Q_1} X_j \xrightarrow{h} X_i \to 0 \tag{1.2}$$

and $\left(F_i^- X'\right)_i = \operatorname{Coker} \widetilde{h} \cong X_i$. Observe also that $\widetilde{h}$ is injective by definition, hence X' is indecomposable. Thus, we proved the first part of the theorem.

For the case of $-$-admissible vertex, define D: Rep $Q \to$ Rep Q^{op}, where Q^{op} is the quiver with all arrows of Q reversed, $D(X_j) = X_j^*$, $D(\rho_\gamma) = \rho_\gamma^*$, and note that $D \circ F_i^+ = F_i^- \circ D$. Since D reverses all maps and change Ker to Coker, the second statement of the theorem follows immediately. $\square$

Note that for an arbitrary X the sequence (1.2) is not exact but $\widetilde{h}$ is injective and $h \circ \widetilde{h} = 0$. Therefore one can define a natural injection $\phi : F_i^- F_i^+ X \to X$, where $\phi_j = \mathrm{id}$ for all $j \neq i$ and ϕ_i coincides with h restricted to $\operatorname{Coker} \widetilde{h}$. In the similar way one can define the natural surjection $\psi : X \to F_i^+ F_i^- X$ if i is a $-$-admissible vertex.

Finally let $Q' = \sigma_i(Q)$, X be a representation of Q' and Y be a representation of Q, $X' = F_i^- X$ and $Y' = F_i^+ Y$. Let $\eta \in \operatorname{Hom}_Q(X, Y')$, define $\chi \in \operatorname{Hom}_{Q'}(X', Y)$ by putting $\chi_j = \mathrm{Id}$ for $j \neq i$ and obtaining χ_i from following commutative diagram

$$\begin{array}{ccccccccc} & & X_i & \xrightarrow{\widetilde{h}} & \oplus X_j & \xrightarrow{h} & X_i' & \to & 0 \\ & & \downarrow^{\eta_i} & & \downarrow^{\oplus \eta_j} & & \downarrow^{\chi_i} & & \\ 0 & \to & Y_i' & \xrightarrow{\widetilde{h}} & \oplus Y_j' & \xrightarrow{h} & Y_i & & \end{array}$$

Note χ_i is uniquely determined by η. In the same way for each $\chi \in \operatorname{Hom}_Q(X', Y)$ one can define $\eta \in \operatorname{Hom}_{Q'}(X, Y')$. A routine check now proves the following

Lemma 1.2. *Let $Q' = \sigma_i(Q)$, X be a representation of Q' and Y be a representation of Q, then*

$$\operatorname{Hom}_{Q'}\left(F_i^- X, Y\right) \cong \operatorname{Hom}_Q\left(X, F_i^+ Y\right).$$

2. Reflection functors and change of orientation.

Lemma 2.1. *Let Γ be a connected graph without cycles, Q and Q' be two quivers on the same graph. Then there exists an enumeration of vertices such that $Q' = \sigma_k \circ \cdots \circ \sigma_1(Q)$ and i is a $+$-admissible vertex for $\sigma_{i-1} \circ \cdots \circ \sigma_1(Q)$.*

Proof. It is sufficient to prove the statement for two quivers Q and Q' different at one arrow. So let $\gamma \in Q_1$. After removing γ, Q splits in two connected components; let Q'' be the component which contains $t(\gamma)$. Enumerate vertices of Q'' in such a way that if $i \to j \in Q''_1$, then $i > j$. This is possible since Q'' does not have cycles. Check that $Q' = \sigma_k \circ \cdots \circ \sigma_1 (Q)$ (here k is the number of all vertices in Q'') and i is a +-admissible vertex for $\sigma_{i-1} \circ \cdots \circ \sigma_1 (Q)$. □

Theorem 2.2. *Let i be a +-admissible vertex for Q and $Q' = \sigma_i (Q)$. Then F_i^+ and F_i^- establish a bijection between indecomposable representations of Q (non-isomorphic to L_i) and indecomposable representations of Q'(non-isomorphic to L_i.)*[1]

Theorem 2.2 follows from 1.1. Together with Lemma 2.1 it allows to change an orientation on a quiver if the quiver does not have cycles.

3. Weyl group and reflection functors.

Given any graph Γ, one can associate with it a certain linear group, which is called a Weyl group of Γ. We denote by $\alpha_1, \dots, \alpha_n$ vectors in the standard basis of $\mathbb{Z}^{\Gamma_0} = \mathbb{Z}^n$, α_i corresponds to the vertex i. These vectors are called simple roots. For each simple root α_i put

$$r_i (x) = x - \frac{2 (x, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i .$$

One can check that r_i preserves the scalar product and $r_i^2 = id$. The linear transformation r_i is called a *simple reflection.* If Γ has no loops, r_i also preserves the lattice generated by simple roots. Hence r_i maps roots to roots. If Γ is Dynkin, the scalar product is positive-definite, and r_i is a reflection in the hyperplane orthogonal to α_i. The *Weyl group* W is a group generated by $r_1, \dots, r_n$. For a Dynkin diagram W is finite (since the number of roots is finite).

Example. Let $\Gamma = A_n$. Let $\varepsilon_1, \dots, \varepsilon_{n+1}$ be an orthonormal basis in $\mathbb{R}^{n+1}$. Then one can take the roots of Γ to be $\varepsilon_i - \varepsilon_j$, simple roots to be $\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \dots, \varepsilon_n - \varepsilon_{n+1}$, $r_i (\varepsilon_j) = 0$ if $j \neq i, i+1$, and $r_i (\varepsilon_i) = \varepsilon_{i+1}$. Therefore W is isomorphic to the permutation group S_{n+1}.

One can check by direct calculation, that (1.1) implies

Lemma 3.1. *If X is an indecomposable representation of Q and* $\dim X = x \neq \alpha_i$, *then* $\dim F_i^{\pm} X = r_i (x)$.

An element $c = r_1 \dots r_n \in W$ is called a *Coxeter transformation.* It depends on the enumeration of simple roots.

Example. In the case $\Gamma = A_n$ a Coxeter element is always a cycle of length $n+1$.

Lemma 3.2. *If $c(x) = x$, then $(x, \alpha_i) = 0$ for all i. In particular for a Dynkin graph $c(x) = x$ implies $x = 0$.*

[1]We will denote by the same letter L_i the representations of quivers with different orientation.

Proof. By definition,

$$c(x) = x + a_1\alpha_1 + \cdots + a_n\alpha_n,\ a_i = -\frac{2(\alpha_i, x + a_1\alpha_1 + \ldots a_{i-1}\alpha_{i-1})}{(\alpha_i, \alpha_i)}.$$

The condition $c(x) = x$ implies all $a_i = 0$. Hence $(x, \alpha_i) = 0$ for all i. □

4. Coxeter functor.

Let Q be a graph without oriented cycles. We call an enumeration of vertices admissible if $i > j$ for any arrow $i \to j$. Such an enumeration always exists. One can easily see that every vertex i is a $+$-admissible for $\sigma_{i-1} \circ \cdots \circ \sigma_1(Q)$ and $-$-admissible for $\sigma_{i+1} \circ \cdots \circ \sigma_n(Q)$. Furthermore,

$$Q = \sigma_n \circ \sigma_{n-1} \circ \cdots \circ \sigma_1(Q) = \sigma_1 \circ \cdots \circ \sigma_n(Q).$$

Define Coxeter functors

$$\Phi^+ = F_n^+ \circ \cdots \circ F_2^+ \circ F_1^+,\ \Phi^- = F_1^- \circ F_2^- \circ \cdots \circ F_n^-.$$

Lemma 4.1. (1) *$Hom_Q(\Phi^- X, Y) \cong \mathrm{Hom}_Q(X, \Phi^+ Y)$;*
(2) *If X is indecomposable and $\Phi^+ X \neq 0$, then $\Phi^-\Phi^+ X \cong X$;*
(3) *If X is indecomposable of dimension x and $\Phi^+ X \neq 0$, then $\dim \Phi^+ X = c(x)$;*
(4) *If Q is Dynkin, then for any indecomposable X there exists k such that $(\Phi^+)^k X = 0$.*

Proof. (1) follows from Lemma 1.2, (2) follows from Theorem 1.1, (3) follows from Lemma 3.1. Let us prove (4). Since W is finite, c has finite order h. It is sufficient to show that for any x there exists k such that $c^k(x)$ is not positive. Assume that this is not true. Then $y = x + c(x) + \cdots + c^{h-1}(x) > 0$ is c invariant. Contradiction with Lemma 3.2.

□

Lemma 4.2. *$\Phi^\pm$ does not depend on a choice of admissible enumeration.*

Proof. Note that if i and j are disjoint and both $+(-)$-admissible, then $F_i^+ \circ F_j^+ = F_j^+ \circ F_i^+ \left(F_i^- \circ F_j^- = F_j^- \circ F_i^-\right)$. If a sequence $i_1, \ldots, i_n$ gives another admissible enumeration of vertices, and $i_k = 1$, then 1 is disjoint with $i_1, \ldots, i_{k-1}$, hence

$$F_1^+ \circ F_{i_{k-1}}^+ \circ \cdots \circ F_{i_1}^+ = F_{i_{k-1}}^+ \circ \cdots \circ F_{i_1}^+ \circ F_1^+.$$

Now proceed by induction. Similarly for Φ^-. □

In what follows we always assume that an enumeration of vertices is admissible.

Corollary 4.3. *Let Q be a Dynkin quiver, X be an indecomposable representation of dimension x, and k be the minimal number such that $c^{k+1}(x)$ is not positive. There exists a unique vertex i such that*

$$x = c^{-k} r_1 \ldots r_{i-1}(\alpha_i),\ X \cong \left(\Phi^-\right)^k \circ F_1^- \circ \cdots \circ F_{i-1}^-(L_i).$$

In particular, x is a positive root and for each positive root x, there is a unique (up to an isomorphism) indecomposable representation of dimension x.

Proof. Follows from Theorem 1.1 and Lemma 3.1. □

5. Further properties of Coxeter functors

Here we assume again that Q is a quiver without oriented cycles and the enumeration of vertices is admissible. We discuss the properties of the bilinear form $\langle , \rangle$. Since we plan to change an orientation of Q we use a subindex $\langle , \rangle_Q$, where it is needed to avoid ambiguity.

Lemma 5.1. *Let i be a $+$-admissible vertex, $Q' = \sigma_i(Q)$, and $\langle , \rangle_Q$, $\langle , \rangle_{Q'}$ the corresponding bilinear forms. Then*

$$\langle r_i(x), y\rangle_{Q'} = \langle x, r_i(y)\rangle_Q .$$

Proof. It suffices to check the formula for a subquiver containing i and all its neighbors. Let $x' = r_i(x)$ and $y' = r_i(y)$. Then

$$x'_i = -x_i + \sum_{i\neq j} x_j,\ y'_i = -y_i + \sum_{i\neq j} y_j,$$

$$\langle x', y\rangle_{Q'} = x'_i y_i - x'_i \sum_{i\neq j} y_j + \sum_{i\neq j} x_j y_j = -x'_i y'_i + \sum_{i\neq j} x_j y_j,$$

$$\langle x, y'\rangle_Q = x_i y'_i - y'_i \sum_{i\neq j} x_j = -x'_i y'_i + \sum_{i\neq j} x_j y_j.$$

□

Corollary 5.2. *For a Coxeter element c we have*

$$\left\langle c^{-1}(x), y\right\rangle = \langle x, c(y)\rangle .$$

If $\Phi^+(Y) \neq 0, \Phi^-(X) \neq 0$, then

$$\dim \mathrm{Ext}^1\left(X, \Phi^+(Y)\right) = \dim \mathrm{Ext}^1\left(\Phi^-(X), Y\right).$$

Proof. First statement follows directly from Lemma 5.1. The second statement follows from the first statement, Lemma 1.2 and the identity

$$\langle x, y\rangle_Q = \dim \mathrm{Hom}_Q(X, Y) - \dim \mathrm{Ext}^1(X, Y) .$$

□

Let $A = k(Q)$ be the path algebra. Recall that any indecomposable projective module is isomorphic to Ae_i.

Lemma 5.3. $F_i^+ \circ \cdots \circ F_1^+ (Ae_i) = 0$, $F_{i-1}^+ \circ \cdots \circ F_1^+ (Ae_i) \cong L_i$.

Proof. One can check by direct calculation that for each component e_jAe_i, $e_jAe_i = 0$ for $j > i$, and

$$F_k^+ \circ \dots \circ F_1^+ (e_jAe_i) = e_jAe_i \text{ for } k < j,\ F_j^+ \circ \dots \circ F_1^+ (e_jAe_i) = 0.$$

□

Corollary 5.4. *$\Phi^+ (P) = 0$ for any projective module P. For any indecomposable projective Ae_i we have*

$$Ae_i = F_1^- \circ \dots \circ F_{i-1}^- (L_i) . \tag{5.1}$$

Proof. The first statement follows from Lemma 5.3 immediately. For the second use Theorem 1.1 and Lemma 5.3.

□

An injective module is a module I such that for any injective homomorphism $i : X \to Y$ and any homomorphism $\varphi : X \to I$, there exists a homomorphism $\psi : Y \to I$ such that $\varphi = \psi \circ i$. A module I is injective iff $\mathrm{Ext}^1 (X, I) = 0$ for any X. One can see analogy with projective modules, however in general there is no nice description of injective (like a summand of a free module).

Exercise. Check that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

In case when A is a finite-dimensional algebra, injective modules are easy to describe. Indeed, the functor $D : A - \mod \to \mod - A$ such that $D (X) = X^*$ maps left projective modules to right injective and vice versa. Therefore any indecomposable injective module is isomorphic to $(e_jA)^*$. Since $D \circ \Phi^+ = \Phi^- \circ D$, one can see easily that $\Phi^- (I) = 0$ for any injective module I. Moreover, one can prove similarly to the projective case that

$$(e_jA)^* \cong F_n^+ \circ \dots \circ F_{j+1}^+ (L_j) .$$

Let $P (j) = Ae_j$ and $I (j) = (e_jA)^*$ and $p (j) = \dim P (j)$, $i (j) = \dim I (j)$. Then

$$c (p (j)) = r_n \dots r_1 (p (j)) = r_n \dots r_{j+1} (-\alpha_j) = -i (j) . \tag{5.2}$$

Note that $\mathrm{Ext}^1 (Ae_j, X) = 0$ for any X and $\dim \mathrm{Hom}_Q (Ae_j, X) = x_j$. Hence

$$\langle p (j) , x\rangle = x_j. \tag{5.3}$$

On the other hand, $\mathrm{Ext}^1 (X, (e_jA)^*) = 0$ and

$$\mathrm{Hom}_Q (X, (e_jA)^*) \cong \mathrm{Hom}_Q (e_jA, X^*) ,$$

which implies $\dim \mathrm{Hom}_Q (X, (e_jA)^*) = x_j$. Thus, we obtain

$$\langle x, i (j)\rangle = x_j. \tag{5.4}$$

Combine together (5.2), (5.3), (5.4) and get

$$\langle p (j) , x\rangle + \langle x, c (p (j))\rangle = 0.$$

Since $p(1), \dots, p(n)$ form a basis, the last equation implies that for arbitrary x and y

$$\langle y, x\rangle + \langle x, c(y)\rangle = 0. \tag{5.5}$$

6. Affine root system

Let Γ be an affine Dynkin graph. Then the kernel of bilinear symmetric form in $\mathbb{Z}^n$ is one-dimensional and generated by

$$\delta = a_0\alpha_0 + a_1\delta_1 + \cdots + a_n\delta_n.$$

We assume without loss of generality that the vertex α_0 is such that $a_0 = 1$. By removing 0 from Γ we get a Dynkin graph which we denote by Γ^0. In affine case roots can be of two kinds: *real*, if $q(\alpha) = 1$, or *imaginary*, $q(\alpha) = 0$.

Lemma 6.1. *Imaginary roots are all proportional to δ, real roots can be written as $\alpha + m\delta$ for some root δ of Γ^0. Every real root can be obtained from a simple root by the action of the Weyl group W.*

Proof. The first statement is obvious, the second follows from the fact that $q(\alpha) = q(\alpha + m\delta)$, hence the projection on the hyperplane generated by $\alpha_1, \dots \alpha_n$ maps a root to a root. To prove the last statement, note that r_i maps every positive root different from α_i to a positive root. Let α be a positive real root, $\alpha = a_0\alpha_0 + \cdots + a_n\alpha_n$, and $h(\alpha) = a_0 + a_1 + \cdots + a_n$. Then $(\alpha, \alpha_i) > 0$ at least for one i. But then $h(r_i(\alpha)) < h(\alpha)$. Thus, one can decrease $h(\alpha)$ by application of simple reflection. In the end one can get a root of height 1, which is a simple root. Similarly for negative roots. $\square$

7. Kronecker quiver

In this section we use Coxeter functors to classify indecomposable representation of the quiver $\hat{A}_1 = \bullet \rightrightarrows \bullet$. The admissible enumeration of vertices is $1 \rightrightarrows 0$, $\delta = \alpha_0 + \alpha_1$. Positive real roots are

$$m\alpha_1 + (m+1)\alpha_0 = -\alpha_1 + (m+1)\delta,\ (m+1)\alpha_1 + m\alpha_0 = \alpha_1 + m\delta,\ m \geq 0.$$

The Coxeter element $c = r_1 r_0$ satisfies

$$c(\alpha_1) = \alpha_1 + 2\delta,\ c(\delta) = \delta.$$

Let $x = m\alpha_1 + l\delta$. If $m > 0$ then $c^{-s}(x)$ is not positive for sufficiently large s. Hence if X is indecomposable of dimension x, then $(\Phi^-)^s X = 0$. If $m < 0$, then $(\Phi^+)^s X = 0$. Thus if $m \neq 0$, then as in the case of Dynkin quiver, X can be obtained from some L_i by application of reflection functor. In particular, we obtain that the dimension of an indecomposable representation is always a root and if this root is real, then the indecomposable with this dimension is unique up to an isomorphism. Indeed, we have either

$$k^m \rightrightarrows^{A}_{B} k^{m+1},$$

where $A = (1_m, 0)\,, B = (0, 1_m)$, or

$$k^{m+1} \rightrightarrows^{C}_{D} k^m,$$

where $C = A^t, D = B^t$.

Classification of indecomposables of dimension $m\delta$ is equivalent to classification of pairs of linear operators $(A, B) : k^m \to k^m$ up to equivalence $(A, B) \sim (PAQ^{-1}, PBQ^{-1})$. Assume that A is invertible, then one may assume that $A = \mathrm{Id}$, and then classify B up to conjugation. Indecomposability of the representation implies that B is equivalent to the Jordan block with some eigenvalue μ. Denote the corresponding representation by ρ_μ. If B is invertible, then A is equivalent to a Jordan block. Denote such representation by σ_μ. One can see that ρ_μ is isomorphic to $\sigma_{\mu^{-1}}$ if $\mu \neq 0$. Now let us prove that at least one of A and B is invertible. Indeed, indecomposability implies that $\operatorname{Ker} A \cap \operatorname{Ker} B = 0$. Hence $A + tB$ is invertible for some t. Without loss of generality one can assume that $A + tB = id$. But then either A or B must be invertible. Thus, we proved that indecomposable representation of dimension $(m, m) = \delta$ are parameterized by a projective line.

For other affine quivers, the situation is more complicated, as there are real roots which remain positive under Coxeter transformation. For example consider the quiver $\widehat{D}_4$

$$\begin{array}{ccccc} & & 5 & & \\ & & \downarrow & & \\ 1 & \to & 2 & \to & 4 \\ & & \uparrow & & \\ & & 3 & & \end{array}$$

Then $c\,(\alpha_1 + \alpha_2 + \alpha_3) = \alpha_4 + \alpha_2 + \alpha_5$, $c^2\,(\alpha_1 + \alpha_2 + \alpha_3) = 0$.