

FINAL EXAM
MATH 252

Choose and solve one problem from the list below. The exam is due December 14.

Problem 1. Let $\mathrm{PSL}_2(\mathbb{F}_q)$ be the quotient of $\mathrm{SL}_2(\mathbb{F}_q)$ by the center. Compute the table of irreducible characters for the group $\mathrm{PSL}_2(\mathbb{F}_q)$ over \mathbb{C} .

Problem 2. Let k be algebraically closed of characteristic 0, G be a finite group, K be a subgroup. The Hecke algebra $H(G, K)$ is the subalgebra of the group algebra $k(G)$

$$H(G, K) = \{u \in k(G) \mid h_1 u h_2 = u \text{ for any } h_1, h_2 \in H\}.$$

(a) Show that $\dim H(G, K)$ equals the number of double cosets $K \backslash G / K$.

(b) Show that $H(G, K)$ is commutative iff the multiplicity of any irreducible representation of G in $\mathrm{Ind}_K^G(\mathrm{triv})$ is not greater than 1.

(c) Prove that $H(G, K)$ is commutative for $G = S_{p+q}$, $K = S_p \times S_q$.

(d) Describe $H(G, K)$ for $G = \mathrm{GL}_3(\mathbb{F}_q)$, K being the subgroup of upper triangular matrices.

Problem 3. Let G be the group of symmetries of an n -dimensional cube in \mathbb{R}^n .

(a) Classify irreducible representation of G . Hint: show that G is a semi-direct product of S_n and the normal subgroup H , isomorphic to Z_2^n , and use induction from H .

(b) Let ρ be the permutation representation of G induced by the action of G on the set of vertices of an n -dimensional cube. Decompose ρ into the sum of irreducibles.

Problem 4.

(a) Let R be a ring, S be a subring. Show that if P is a projective S -module, then $\mathrm{Ind}_S^R P = R \otimes_S P$ is a projective R -module.

(b) Let p be a prime number. Describe projective indecomposable and irreducible representations of S_p over \mathbb{F}_p . Hint: use induction from S_{p-1} .

(c) For $p = 5$ find dimensions of all irreducible representations.

Problem 5. Show that the Coxeter functor Φ^+ coincides with the functor τ defined in Crawley-Boevey lectures (page 22), i.e. $\Phi^+(X) = \mathrm{DExt}^1(X, k(Q))$ for any X .

Problem 6. Choose your favorite affine quiver different from \hat{A}_n (if you choose \hat{D}_n do it for all n) and your favorite orientation. Describe all indecomposable representations, regular simple representations and tubes.