

## THE CHARACTER OF AN IRREDUCIBLE $GL_n$ -MODULE.

### 1. INVARIANT INTEGRATION ON $U(n)$

Let

$$G = U(n) = \{X \in GL_n(\mathbb{C}) \mid X\bar{X}^t = 1_n\}$$

be a compact subgroup in  $GL_n(\mathbb{C})$ . The tangent space to the identity  $T_e(G)$  coincides with the subspace of skew-Hermitian matrices ( $\bar{Y}^t = -Y$ ) and the adjoint representation is given by the formula

$$\text{Ad}_X(Y) = XYX^{-1}.$$

**Exercise.** Check that  $\det \text{Ad}_X = 1$  for any  $X \in G$ .

Let  $T$  be the abelian subgroup of diagonal matrices:

$$T = \{z = \text{diag}(z_1, \dots, z_n \mid |z_i| = 1)\}.$$

By  $dz$  we denote the invariant volume form  $T$ . If  $z_j = e^{i\theta_j}$ , then  $dz = \frac{1}{(2\pi)^n} d\theta_1 \cdots d\theta_n$ .

Since every unitary operator has an orthonormal eigen basis, every conjugacy class in  $G$  intersects  $T$ .

**Lemma 1.1.** *Let  $f(g)$  be a continuous class function on  $G$ . Then*

$$\int f(g)dg = \frac{1}{n!} \int f(z) \prod_{i < j} (z_i - z_j)(\bar{z}_i - \bar{z}_j) dz.$$

*Proof.* Consider the map  $\Phi : G/T \times T \rightarrow G$  defined by the formula

$$\Phi(g, z) = gzg^{-1}.$$

Note that  $\Phi$  is surjective. Moreover, if  $G_r$  denote the open subset of unitary matrices without multiple eigenvalues, then for any  $g \in G_r$  the fiber  $\Phi^{-1}(g)$  is a finite set of cardinality  $n!$ . Indeed, if  $z_1, \dots, z_n$  are eigen values of  $g$ , then

$$\Phi^{-1}(g) = \{u_s \text{diag}(z_{s(1)}, \dots, z_{s(n)} u_s^{-1} \mid s \in S_n, u_s \in G/T\}.$$

Let  $du$  be an invariant form on  $G/T$ . We have

$$(1.1) \quad \int f(g)dg = \frac{1}{n!} \int f(u, z) \Phi_{(u,z)}^* du dz.$$

To calculate  $\Phi_{(u,z)}^* du dz$  consider smooth functions  $z(s) \in T$  and  $u(s) \in G/T$  such that  $z(0) = e$  and  $u(0) = e$ . Then

$$\frac{d}{ds} \Phi(u, z(s))|_{s=0} = \frac{d}{ds} (uzu^{-1}uz(s)u^{-1})|_{s=0} = \Phi(u, z) \text{Ad}_u(z'(0)),$$

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and

$$\frac{d}{ds}\Phi(uu(s), z)|_{s=0} = \frac{d}{ds}(uu(s)zu(s)^{-1}u^{-1})|_{s=0} = \Phi(u, z) \text{Ad}_u(\text{Ad}_{z^{-1}} - 1)u'(0).$$

Thus,

$$\Phi_{(u,z)}^* dudz = \det \text{Ad}_u(\text{Ad}_{z^{-1}} - 1) dudz.$$

But  $\det \text{Ad}_u = 1$ . To calculate  $\det(\text{Ad}_{z^{-1}} - 1)$  note that the eigenvalues of  $\text{Ad}_z$  in  $T_e(G/T)$  are  $\frac{z_i}{z_j}$  for all  $i \neq j$ . Hence we obtain

$$(1.2) \quad \Phi_{(u,z)}^* dudz = \prod_{i \neq j} \left(\frac{z_i}{z_j} - 1\right) dudz = \prod_{i < j} (z_i - z_j)(z_i^{-1} - z_j^{-1}) dudz.$$

Let  $f(g)$  be a class function, then (1.1) and (1.2)

$$\int f(g) dg = \frac{1}{n!} \int f(z) \prod_{i < j} (z_i - z_j)(z_i^{-1} - z_j^{-1}) dudz.$$

This implies Lemma. □

**Lemma 1.2.** *If  $\lambda = (\lambda_1, \dots, \lambda_n)$  and  $\mu = (\mu_1, \dots, \mu_n)$  are two partitions, then for the Schur polynomials  $S_\lambda(z)$ ,  $S_\mu(z)$  we have the identity*

$$\frac{1}{n!} \int S_\lambda(z^{-1}) S_\mu(z) \prod_{i < j} (z_i - z_j)(z_i^{-1} - z_j^{-1}) dz = \delta_{\lambda, \mu}.$$

*Proof.* An easy consequence of the formula for a Schur polynomial and the fact  $\int f(z) dz$  equals the constant term of  $f(z) \in \mathbb{C}[z_1^\pm, \dots, z_n^\pm]$ . □

## 2. REPRESENTATIONS OF $GL_n(\mathbb{C})$

Recall that  $W_\lambda$  denote the irreducible representation of  $GL_n(\mathbb{C})$  corresponding to a partition  $\lambda$ .

**Lemma 2.1.**  *$W_\lambda$  is an irreducible representation of  $U(n)$ .*

*Proof.* Let  $\rho : GL_n(\mathbb{C}) \rightarrow GL(E^{\otimes p})$  be the natural representation of  $GL_n(\mathbb{C})$ . It is easy to see that the span of  $\rho_g$  for all  $g \in U(n)$  coincides with the span of  $\rho_g$  for all  $g \in GL_n(\mathbb{C})$ . By definition  $W_\lambda$  is a subrepresentation of  $E^{\otimes p}$  for  $p = \lambda_1 + \dots + \lambda_n$ . The statement follows. □

Let  $\chi_\lambda$  denote the character of  $W_\lambda$ . Then the orthogonality relation and Lemma 1.1 imply

$$\frac{1}{n!} \int \chi_\lambda(z^{-1}) \chi_\mu(z) \prod_{i < j} (z_i - z_j)(z_i^{-1} - z_j^{-1}) dz = \delta_{\lambda, \mu}.$$

On the other hand, Problem 4 from homework 7 imply

$$(2.1) \quad \chi_\lambda(z) = z^\lambda + \sum_{\mu < \lambda} c_{\mu, \lambda} z^\mu,$$

where  $<$  denote the lexicographic order.

It follows easily that the formula for  $S_\lambda$  that (2.1) also holds for  $S_\lambda(z)$ .

**Theorem 2.2.**  $\chi_\lambda(z) = S_\lambda(z)$ .

*Proof.* We have two orthonormal systems  $\chi_\lambda$  and  $S_\lambda$  in the space of symmetric polynomials. They both satisfy 2.1. One can easily show by induction in lexicographic order that

$$S_\lambda = \chi_\lambda + \sum_{\mu < \lambda} b_{\mu,\lambda} \chi_\mu.$$

Orthogonality implies all  $b_{\mu,\lambda}$  are zero. □