PROBLEM SET
MATH 261A

1. Let G be a connected simply connected Lie group, $\mathfrak{g}$ be its Lie algebra. Check that the commutator $[G, G]$ is a connected closed subgroup with Lie algebra $[\mathfrak{g}, \mathfrak{g}]$.

2. Let $\mathfrak{g}$ be a semisimple Lie algebra, V be a faithful representation of $\mathfrak{g}$, $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_n\}$ be bases in $\mathfrak{g}$ dual with respect to the form $B_V(X, Y) = \operatorname{tr}_V XY$. The Casimir operator in V is defined by

$$C_V = \sum_{i=1}^{n} e_i f_i.$$

Define the operator γ: $\operatorname{Hom}_k(\Lambda^j \mathfrak{g}, V) \to \operatorname{Hom}_k(\Lambda^{j-1} \mathfrak{g}, V)$ by the formula

$$\gamma c(x_1, \dots, x_{j-1}) = \sum_{i=1}^{n} e_i c(f_i, x_1, \dots, x_{j-1}).$$

Let $d : \operatorname{Hom}_k(\Lambda^{j-1} \mathfrak{g}, V) \to \operatorname{Hom}_k(\Lambda^j \mathfrak{g}, V)$ be the differential in the cohomology complex. Check that

$$\gamma \circ d + d \circ \gamma = C_V.$$

3. Let V be an irreducible non-trivial representation of a semisimple Lie algebra $\mathfrak{g}$. Prove that $H^j(\mathfrak{g}; V) = 0$ for all j. (You have to modify the statement of problem 2 if V is not faithful).

4. Evaluate $H^j(\mathfrak{g}; k)$ for $\mathfrak{g} = \mathrm{sl}_2(k)$.

5. Let $\mathfrak{g}$ be a simple Lie algebra over algebraically closed field. Check that two invariant forms on $\mathfrak{g}$ are proportional.

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