PROBLEM SET # 12
MATH 249

Due November 30.

1. Let $s_\lambda(x_1, \dots, x_N)$ be the Schur polynomial, $\bar{\lambda} = \lambda + \delta$, where $\delta = (N-1, \dots, 1, 0)$. Show that

$$s_\lambda(1, \dots, 1) = \prod_{i<j\leq N} \frac{(\bar{\lambda}_i - \bar{\lambda}_j)}{(j-i)}.$$

Hint: use the Weyl formula $s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}$.

2. Let f_λ denote the number of standard tableaux of shape $\lambda = (\lambda_1, \dots, \lambda_k)$ and type (1,...,1). Let $|\lambda| = n$, $\delta = (k-1, k-2, \dots, 0)$ and $\bar{\lambda} = \lambda + \delta$. Prove that

$$f_\lambda = \frac{n!}{\bar{\lambda}_1! \dots \bar{\lambda}_k!} \prod_{i<j\leq k} (\bar{\lambda}_i - \bar{\lambda}_j).$$

Hint: $f_\lambda = \langle h_1^n, s_\lambda \rangle$. Use the Weyl formula to show that f_λ is the coefficient of $x^{\bar{\lambda}}$ in $a_\delta(x_1 + \dots + x_k)^n$.

3. Using problem 2 prove the famous hook formula

$$f_\lambda = \frac{n!}{\prod \gamma_{ij}},$$

here the product is taken over all cells (i,j) of the diagram λ and γ_{ij} is the length of the hook diagram consisting of (i,j) and all cells to the right of (i,j) in the i-th row and below (i,j) in the j-th column.

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