Practice problems
Solutions

1. Evaluate the integral

$$\int_\gamma \frac{\sin z^2}{z^3} dz,$$

where γ is the unit circle, centered at 0, traveled once counterclockwise.

2. Let $f(z)$ be an entire function. Show that for every $w \in \mathbb{C}$ there exists a sequence z_n such that $\lim f(z_n) = w$. (Hint: assume that there is no such sequence and consider $g(z) = \frac{1}{f(z)-w}$.)

3. Does the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converge uniformly on $\mathbb{C}$? Explain your answer.

4. Write the Taylor series expansion for

$$f(z) = \frac{z}{z^2+2}$$

at $z = 0$, and find the radius of convergence for this series.

5. Find the maximum of $\operatorname{Im}(z^4)$ on the square $[0,1] \times [0,1]$.

Solutions.

1.

$$\int_\gamma \frac{\sin z^2}{z^3} dz = \operatorname{Res}_{z=0} \frac{\sin z^2}{z^3} = 2\pi i$$

2. Suppose that such sequence does not exist. Then there is $\varepsilon > 0$ such that $|f(z) - w| \geq \varepsilon$ for all z. (Indeed, otherwise there exists a sequence z_n such that $|f(z_n) - w| < 1/n$, and this sequence satisfies the condition of the problem.) Therefore $|g(z)| = |\frac{1}{f(z)-w}| \leq 1/\varepsilon$. Thus $g(z)$ is entire and bounded, and by Liouville's theorem it is constant. Then of course $f(z)$ is constant. Contradiction.

3. No. If the series $\sum_{n=0}^{\infty} g_n(z)$ converges uniformly on $\mathbb{C}$, then $g_n(z)$ converges uniformly to zero, i.e. for any $\varepsilon > 0$ there exists N such that $|g_n(z)| < \varepsilon$ for all $n > N$ and all $z \in \mathbb{C}$. In particular $g_n(z)$ is bounded for sufficiently large n. In our case $g_n(z) = \frac{z^n}{n!}$ is unbounded for all n.

4. The radius of convergence is $\sqrt{2}$.

$$\frac{z}{z^2+2} = \frac{z}{2}\frac{1}{1+z^2/2} = \frac{z}{2}\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{2^n} = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{2^{n+1}}$$

5. The function is harmonic, so the maximum is attained on the boundary.

$$\operatorname{Im} z^4 = 4x^3y - 4xy^3.$$

When $x = 0$ or $y = 0$, $\operatorname{Im} z^4 = 0$. When $y = 1$, $0 \leq x \leq 1$, $\operatorname{Im} z^4 = 4x^3 - 4x \leq 0$. When $x = 1$, $0 \leq y \leq 1$, $\operatorname{Im} z^4 = 4y - 4y^3$. To find maximum, find the critical point given by the equation

$$\left(4y - 4y^3\right)' = 4 - 12y^2 = 0.$$

Then $y = \frac{1}{\sqrt{3}}$, and maximum is $\frac{4}{\sqrt{3}} - \frac{4}{3\sqrt{3}} = \frac{8}{3\sqrt{3}}$.