Sample midterm
Math 185

The midterm will take place in 9 Lewis on Tuesday, October 5 during usual class hours. It will be a "closed book exam". It will cover sections 1–45 of the book (first 4 homework assignments). Your score for the first midterm contributes 30% to your total grade. Below is a sample midterm with solutions.

1. Solve the equation

$$\frac{\sin z}{\cos z} = 2i.$$

2. Find the image of $|\operatorname{Re} z| \leq 1$ under the mapping $f(z) = e^z$.

3. For each of the following sets, state whether or not it is a domain. Justify your answer

(a) $|\operatorname{Re} z| < 2$, $|\operatorname{Im} z| < 2$.

(b) $1 \leq |z| < 3$

4. Find $\lim_{z \to 0} \frac{\sin z}{z}$.

5. Show that $f(z) = \bar{z}^2$ is not analytic at any point .

6. Find the integrals

(a) $\int_\gamma y dz$, where γ is the segment from 0 to $3i$.

(b) $\int_\gamma \frac{1}{z^2+1} dz$, where γ is the circle with center i and radius 1.

7. Show that if $f(z)$ is analytic in some domain A and $f(z)$ is real for any $z \in A$, then $f(z)$ is constant on A.

8. Show that the principal value z^{1+i} is analytic in the region $-\pi < \arg z < \pi$ and find its derivative.

9. Is it true for the principle values that

$$\left(z^i\right)^i = z^{-1}?$$

Solutions.

1. Let $u=e^{iz}$. Then the equation becomes

$$\frac{u-u^{-1}}{i\left(u+u^{-1}\right)}=2i.$$

It leads to the equation $u-u^{-1}=-2\left(u+u^{-1}\right)$, which leads to $3u^2=-1$. This equation has two solutions $u=\pm i/\sqrt{3}$. Then $z=-i\log\left(\pm i/\sqrt{3}\right)=\pi/2+\pi n+(\frac{1}{2}\ln 3)i$.

2. Use $|e^z|=e^{\operatorname{Re} z}$. The image is given by $e^{-1}\leq|z|\leq e$.

3.

(a) The set is open and connected. So it is a domain. To show that it is open pick up any point $z=x+yi$ Let ε be the minimum of $|x\pm 2|,|y\pm 2|$. Then the neigborhood of z with radius ε lies entirely in the region. To see that it is connected just show that it is convex.

(b) The set is not open, for example, $z=1$ is not an interior point. So it is not a domain.

4. By the definition of the derivative

$$(\sin z)'\,|_{z=0}=\lim_{z\to 0}\frac{\sin z}{z}=\cos 0=1.$$

5. We will show that $f(z)=\bar{z}^2$ does not satisfy Cauchy-Riemann equation: $u(x,y)=\operatorname{Re}(x-iy)^2=x^2-y^2$, $v(x,y)=-2xy$. Then $\frac{\partial u}{\partial x}=2x\neq\frac{\partial v}{\partial y}=-2x$.

6.

(a)$\int_0^3 td\,(it)=i\int_0^3 tdt=\frac{9i}{2}$

(b) Write $\frac{1}{z^2+1}=\frac{i}{2}\left(\frac{1}{z+i}-\frac{1}{z-i}\right)$. The function $\frac{1}{z+i}$ is analytic inside γ and therefore $\int_\gamma\frac{1}{z+i}dz=0$. The second integral

$$\int_\gamma\frac{1}{z-i}dz=\int_0^{2\pi}\frac{1}{e^{\mathrm{it}}}d\left(i+e^{\mathrm{it}}\right)=2\pi i.$$

Therefore

$$\int_\gamma\frac{1}{z^2+1}dz=\frac{-i}{2}\int_\gamma\frac{1}{z-i}dz=\pi.$$

7. Let $f(z)=u(x,y)+iv(x,y)$. The condition on $f(z)$ implies that $v(x,y)=0$. By Cauchy-Riemann equation $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=0$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=0$. Since the partial derivatives of $v(x,y)$ are zero, then $v(x,y)$ is constant. This implies that f is constant.

8. Use $z^{1+i}=e^{(1+i)\operatorname{Log}(z)}$. The function is analytic in the same region as $\operatorname{Log} z$ because the composition of analytic functions is analytic. The derivative is $(1+i)\,z^i$ by the chain rule.

9. No. Let $z=e^{2\pi}$, then

$$z^i=e^{2\pi i}=1,(z^i)^i=1,z^{-1}=e^{-2\pi}.$$