

PRACTICE MIDTERM # 1
MATH 121B

1. Use Fermat's principle to find the path of a light ray through a medium of index of refraction $r^{-1/2}$.

2. Write in terms of π the following values :

$$(a) \Gamma(3.5); (b) B\left(\frac{5}{6}, \frac{7}{6}\right).$$

3. (a) Find the period T of a pendulum (with a string of length l) for swings from the angle $\frac{\pi}{3}$ to $-\frac{\pi}{3}$ and back in terms of complete elliptic integral.

(b) Let T_0 denote the period of a very small oscillation of the same pendulum. Use the approximation

$$K(k) \approx \frac{\pi}{2} \left(1 + \frac{1}{4}k^2\right)$$

to estimate $\frac{T}{T_0}$.

4. Solve the differential equation

$$y'' = x^2y$$

using power series.

5. Prove the following identities for the Legendre's polynomials

$$P'_l(x) = (2l-1)P_{l-1}(x) + (2l-5)P_{l-3}(x) + \cdots + P_0$$

if l is odd,

$$P'_l(x) = (2l-1)P_{l-1}(x) + (2l-5)P_{l-3}(x) + \cdots + 3P_1$$

if l is even.

Hint: write

$$P'_l(x) = \sum_{k=0}^{l-1} c_k P_k(x)$$

and find c_k using the orthogonality property.

Solutions are on the next page.

Solutions.

1.

$$\int n ds = \int r^{-1/2} \sqrt{1 + r^2 (\theta')^2} d\theta.$$

Euler's equation gives

$$\frac{r^{-1/2} r^2 \theta'}{\sqrt{1 + r^2 (\theta')^2}} = C.$$

Solve it

$$\begin{aligned} \frac{r^{-1/2} r^2 \theta'}{\sqrt{1 + r^2 (\theta')^2}} &= C \\ r^3 (\theta')^2 &= C^2 (1 + r^2 (\theta')^2) \\ (r^3 - C^2 r^2) (\theta')^2 &= C^2 \\ \theta' &= \frac{C}{r \sqrt{r - C^2}} \\ \theta &= \int \frac{C}{r \sqrt{r - C^2}} dr \end{aligned}$$

To take the integral use substitution $u = \sqrt{r - C^2}$. Get

$$\begin{aligned} \theta &= 2 \int \frac{C}{u^2 + C^2} du \\ \theta &= 2 \tan^{-1} \left(\frac{u}{C} \right) + A = 2 \tan^{-1} \left(\frac{\sqrt{r - C^2}}{C} \right) + A. \end{aligned}$$

2.

$$\begin{aligned} \Gamma(3.5) &= \frac{5}{2} \frac{3}{2} \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{15\sqrt{\pi}}{8} \\ B\left(\frac{5}{6}, \frac{7}{6}\right) &= \frac{\Gamma\left(\frac{5}{6}\right) \Gamma\left(\frac{7}{6}\right)}{\Gamma(2)} = \frac{1}{6} \Gamma\left(\frac{5}{6}\right) \Gamma\left(\frac{1}{6}\right) \end{aligned}$$

Now use

$$\begin{aligned} \Gamma(p) \Gamma(1-p) &= \frac{\pi}{\sin(p\pi)} \\ B\left(\frac{5}{6}, \frac{7}{6}\right) &= \frac{1}{6} \frac{\pi}{\sin(\pi/6)} = \frac{\pi}{3} \end{aligned}$$

3.

$$\begin{aligned} T &= 4 \sqrt{\frac{l}{g}} K\left(\sin \frac{\pi}{6}\right) = 4 \sqrt{\frac{l}{g}} K\left(\frac{1}{2}\right) \\ K\left(\frac{1}{2}\right) &\approx \frac{\pi}{2} \left(1 + \frac{1}{16}\right) \end{aligned}$$

$$T_0 \approx 2\pi \sqrt{\frac{l}{g}}$$

$$\frac{T}{T_0} \approx \frac{17}{16}$$

4. Substitute

$$y = \sum_{n=0}^{\infty} a_n x^n$$

into the equation. Get

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n = \sum_{n=2}^{\infty} a_{n-2} x^n.$$

Note that $a_2 = a_3 = 0$ and

$$a_{n+4} = \frac{a_n}{(n+4)(n+3)}.$$

That gives two particular solutions:

$$y_1(x) = 1 + \frac{1}{3 \cdot 4} x^4 + \frac{1}{3 \cdot 4 \cdot 7 \cdot 8} x^8 + \dots$$

$$y_2(x) = x + \frac{1}{4 \cdot 5} x^5 + \frac{1}{4 \cdot 5 \cdot 8 \cdot 9} x^9 + \dots$$

5. Write down

$$P_l'(x) = \sum_{k=0}^{l-1} c_k P_k(x).$$

Here

$$c_k = \frac{2k+1}{2} \int_{-1}^1 P_l'(x) P_k(x) dx = \frac{2k+1}{2} \left(P_l(x) P_k(x) \Big|_{-1}^1 - \int_{-1}^1 P_l(x) P_k'(x) dx \right)$$

Since $k < l$

$$\int_{-1}^1 P_l(x) P_k'(x) dx = 0$$

and

$$P_l(x) P_k(x) \Big|_{-1}^1 = P_l(1) P_k(1) - P_l(-1) P_k(-1) = 1 - (-1)^{k+l}.$$

So we get $c_k = 0$ if $k+l$ is even, $c_k = 2k+1$ if $k+l$ is odd.