Practice midterm

1. A long (infinite) wire occupying the positive part of x-axis is initially at rest. The end $x = 0$ is oscillating up and down (in y-direction) so that

$$y(0,t) = \sin t,\ t > 0.$$

Find the vertical displacement $y(x,t)$.

Hint: you have to solve the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2},$$

with boundary condition

$$y(0,t) = \sin t,\ t > 0$$

and with initial condition

$$y(x,0) = \frac{\partial y}{\partial t}(x,0) = 0,\ x > 0.$$

Look for solution in the form $y(x,t) = f(x+vt) + g(x-vt)$.

2. The surface temperature $u(x,y,z)$ of the sphere $x^2 + y^2 + z^2 = 1$ is held at $u = z^3$. Find the interior temperature $u(x,y,z)$.

3. A disk of radius 1 has initial temperature 50°. Starting at $t = 0$ the circumference of the disk is held at 10°. Find the temperature distribution as a function of time.

4. Solve the Laplace equation for a function u in the ring $1 \le r \le 10$ satisfying the boundary conditions

$$u(x,y) = x \text{ if } r = 1,\ u(x,y) = y \text{ if } r = 10.$$

Solutions.

1. The condition $y(x,0)=0$ implies

$$f(x)+g(x)=0$$

for $x>0$.

The condition $\frac{\partial y}{\partial t}(x,0)=0$ implies

$$f'(x)-g'(x)=0$$

for $x>0$.

Combining these two conditions one gets

$$y(x,t)=f(x+vt)-f(x-vt)$$

for some function $f(x)$ such that $f(x)=0$ for $x>0$. To find $f(x)$ for $x<0$ use boundary condition $y(0,t)=\sin t$ for $t>0$. We have

$$f(vt)-f(-vt)=\sin t.$$

Therefore $f(-vt)=-\sin t$, which implies $f(z)=\sin\frac{z}{v}$ for negative z. Thus, for positive x,

$$y(x,t)=0 \text{ if } x>vt$$

and

$$y(x,t)=-\sin\left(\frac{x}{v}-t\right)=\sin\left(t-\frac{x}{v}\right) \text{ if } x<vt.$$

So every point of the wire starts oscillating at the moment $t=\frac{x}{v}$.

2. Look for solution in the form

$$u=\sum a_n r^n P_n(\cos\theta),$$

for $r=1$ we have

$$\sum a_n P_n(\cos\theta)=\cos^3\theta.$$

Then we have $a_3=\frac{2}{5}$, $a_1=\frac{3}{5}$, all other coefficients are zero. So we have

$$u=\frac{3}{5}r\cos\theta+\frac{2}{5}r^3\left(\frac{5}{2}\cos^3\theta-\frac{3}{2}\cos\theta\right).$$

In Cartesian coordinates

$$u=\frac{3}{5}z+\frac{2}{5}z^3-\frac{3}{5}z\left(x^2+y^2\right).$$

3. Write down the heat equation

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=\frac{1}{\alpha^2}\frac{\partial u}{\partial t}.$$

We have the following boundary and initial conditions

$$u(1,\theta,t)=10,\ u(r,\theta,0)=50,\ u(r,\theta,\infty)=10.$$

Clearly u does not depend on θ. After separation of variables we get

$$u = R(r)\,T(t)\,,\ \frac{1}{\alpha^2}T' = -k^2 T,\ \frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} = -k^2 R.$$

Solving we get

$$T = e^{-k^2\alpha^2 t},\ R = J_0(kr)\,.$$

We look for solution in the form

$$u = 10 + \sum a_m J_0(k_m r)\, e^{-k_m^2\alpha^2 t},$$

where $k_1, k_2, \ldots$ are zeros of J_0. The coefficients are given by the formula

$$a_m = \frac{2}{J_1^2(k_m)}\int_0^1 40 r J_0(k_m r)\,dr = \frac{80}{J_1^2(k_m)}\frac{J_1(k_m)}{k_m} = \frac{80}{k_m J_1(k_m)}$$

(Check similar calculation in Section 13.5.)

4. Look for solution in the form

$$u = \sum \left(a_n r^n + b_n r^{-n}\right)\cos n\theta + \left(c_n r^n + d_n r^{-n}\right)\sin n\theta + c\ln r.$$

When $r = 1$ we have

$$\sum (a_n + b_n)\cos n\theta + (c_n + d_n)\sin n\theta = \cos\theta.$$

That implies

$$a_n + b_n = 0 \text{ for } n > 1,\ c_n + d_n = 0,\ a_1 + b_1 = 1.$$

When $r = 10$ we have

$$\sum \left(a_n 10^n + b_n 10^{-n}\right)\cos n\theta + \left(c_n 10^n + d_n 10^{-n}\right)\sin n\theta + c\ln 10 = 10\sin\theta.$$

That gives

$$a_n 10^n + b_n 10^{-n} = 0,\ c_n 10^n + d_n 10^{-n} = 0 \text{ for } n > 1,\ 10c_1 + \frac{d_1}{10} = 10,\ c = 0.$$

Therefore

$$a_n = b_n = c_n = d_n = 0 \text{ for } n > 1,\ a_1 = -\frac{1}{99},\ b_1 = \frac{100}{99},\ c_1 = \frac{100}{99},\ d_1 = -\frac{100}{99}.$$

$$u = -\frac{r}{99}\cos\theta + \frac{100r^{-1}}{99}\cos\theta + \frac{100}{99}\left(r - r^{-1}\right)\sin\theta.$$