SAMPLE FINAL EXAM
MATH $121B$

1. A particle slides without a friction around a vertical cylinder under the force of gravity. Set up the Lagrange equation.

2. Let a and b be two critical points of the Legendre's polynomial $P_l(x)$ on interval $[-1,1]$. Prove that if $l>0$ then

$$\int_a^b P_l(x)\,dx=0.$$

3. Using power series show that the equation

$$\left(1-x^2\right)y''-xy'+m^2y=0$$

has a polynomial solution for every integer m.

4. (a) Find characteristic frequencies and normal modes of a square membrane with side a.

(b) Find all simple combinations of normal modes which have nodal lines at the diagonal $y=x$.

(c) Use (b) to find characteristic frequencies of a membrane in the form of a right triangle with angle 45°.

5. A string has initial displacement $y_0=x(l-x)$. Find the displacement as a function of x and t.

6. Find a steady state temperature in a semi-infinite covering the region $0\le y\le 1$, $x\ge 0$, if the temperature along the x and y-axis is zero, and the top edge is kept at

$$u(x,1)=\begin{cases}100x, & x<1\\ 0, & x>1\end{cases}.$$

7. Find the Green function for the Laplace equation in the region $x>0, y>0$ in the plane with boundary condition $u(x,0)=u(0,y)=0$.

8. Five letters are put randomly into five envelopes (each envelope contains exactly one letter). What is the probability that at least one letter gets into the correct envelope?

9. A 300-page book has, on the average, one misprint every 5 pages. On about how many pages would you expect to find

(a) no misprints,

(b) one misprint,

(c) two misprints?

10. 5 identical balls are randomly put in 10 boxes. What is the probability that all 5 balls got in the first five boxes for

(a) Maxwell-Boltzmann statistics,

(b) Fermi-Dirac statistics,

(c) Bose-Einstein statistics?

In which case probability is the biggest, the smallest?

11. A multiple-choice test has 100 questions. Each question has four choices for the answer, and only one choice is correct. Suppose that you do not know the subject and choose all your answers randomly. Using Chebyshev inequality estimate the probability that you score for the exam is more than 50%.

Solutions.

1.

$$L = \frac{mv^2}{2} - mgz.$$

In cylindrical coordinates

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2.$$

In our situation r is fixed. So

$$v^2 = r^2 \left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2,$$

$$L = \frac{m}{2}\left(r^2 \left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right) - mgz.$$

The equations are

$$\frac{d\theta}{dt} = \text{const}\,, \; \frac{d^2 z}{dt^2} = -g.$$

2. If $y = P_l(x)$, then

$$l(l+1)\, y = 2xy' + \left(x^2 - 1\right) y''.$$

$$\int_a^b y dx = \frac{1}{l(l+1)} \int_a^b 2xy' + \left(x^2 - 1\right) y''\, dx =$$

$$\frac{1}{l(l+1)} \int_a^b d\left(\left(x^2 - 1\right) y'\right) = \frac{1}{l(l+1)} \left(b^2 - 1\right) y'(b) - \left(a^2 - 1\right) y'(a) = 0.$$

3. Look for solution in the form

$$y = \sum a_n x^n.$$

Then we have the following equations on the coefficients

$$(n+2)(n+1)\, a_{n+2} - n(n-1)\, a_n - n a_n + m^2 a_n = 0,$$

$$a_{n+2} = \frac{\left(n^2 - m^2\right) a_n}{(n+2)(n+1)}.$$

If m is even, take $a_0 = 1$, $a_2 = \frac{-m^2}{2}, \ldots, a_{m+2} = 0$. If m is odd, take $a_1 = 1$, $a_3 = \frac{1-m^2}{6}, \ldots, a_{m+2} = 0$. In both cases we have a polynomial solution

4. (a) Characteristic frequencies are

$$\frac{v}{2a}\sqrt{n^2 + m^2}.$$

(b) Normal modes of vibrations with nodal line $y = x$ are

$$\left(\sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi m y}{a}\right) - \sin\left(\frac{\pi n y}{a}\right) \sin\left(\frac{\pi m x}{a}\right)\right) \sin \frac{\pi v t \sqrt{n^2 + m^2}}{a},$$

$$\left(\sin\left(\frac{\pi nx}{a}\right)\sin\left(\frac{\pi my}{a}\right)-\sin\left(\frac{\pi ny}{a}\right)\sin\left(\frac{\pi mx}{a}\right)\right)\cos\frac{\pi vt\sqrt{n^2+m^2}}{a},$$

(c) characteritstic frequencies are

$$\frac{v}{2a}\sqrt{n^2+m^2}, \text{where}\, m>n.$$

5. According to formula obtained in class

$$y(x,t)=\frac{y_0(x+vt)+y_0(x-vt)}{2},$$

where $y_0(x)$ extends to the whole line by condition that it is periodic with period $2l$ and odd. To obtain a formula for $y(x,t)$ we may assume that $x<\frac{l}{2}$ by symmetry of y_0, and $t<\frac{l}{2v}$. Then we have

$$y(x,t)=\begin{cases} x(l-x)-v^2t^2, & x>vt \\ lx-2xvt, & x<vt \end{cases}.$$

6. Look for solution in the form

$$u(x,y)=\int_0^\infty B(k)\sin kx\sinh kydk.$$

Then $u(0,y)=0$, $u(x,0)=0$. To find $B(k)$ use

$$u(x,1)=\int_0^\infty B(k)\sin kx\sinh kdk.$$

By inverse sin Fourier transform formula

$$B(k)\sinh k=\frac{2}{\pi}\int_0^\infty u(x,1)\sin kxdx,$$

$$\int_0^\infty u(x,1)\sin kxdx=\int_0^1 100x\sin kxdx=$$

$$=100\frac{-kx\cos kx+\sin kx}{k^2}\Big|_0^1=100\frac{-k\cos k+\sin k}{k^2},$$

$$B(k)=\frac{-200k\cos k+200\sin k}{k^2\pi\sinh k}.$$

7. Use the fact that

$$V(x,y,x_0,y_0)=\frac{1}{4\pi}\ln\left((x-x_0)^2+(y-y_0)^2\right)$$

satisfies the equation

$$\nabla^2V=\delta(x-x_0,y-y_0).$$

To satisfy the boundary conditions set

$$G(x,y,x_0,y_0)=V(x,y,x_0,y_0)-V(x,y,-x_0,y_0)-V(x,y,x_0,-y_0)+V(x,y,-x_0,-y_0).$$

8. The probability of each distribution of letters in envelopes is $\frac{1}{5!}$, the probability that the first letter goes to the right envelope is $\frac{4!}{5!}$, the first and the second go to the right envelope is $\frac{3!}{5!}$, the first, the second and the third letter goes to the right envelope is $\frac{2!}{5!}$ e.t.c. By inclusion exclusion formula we obtain

$$P = 5\frac{4!}{5!} - 10\frac{3!}{5!} + 10\frac{2!}{5!} - 5\frac{1}{5!} + \frac{1}{5!} = \frac{19}{30}$$

9. Use Poisson distribution

$$P_n = \frac{\mu^n}{n!}e^{-\mu}.$$

In our case $\mu = \frac{1}{5} = 0.2$.

$$P_0 = e^{-0.2},\ P_1 = 0.2e^{-0.2},\ P_2 = 0.02e^{-0.2}.$$

If n_k is the expected number of pages with k misprints, then $n_k = 300P_k$. So $n_0 \approx 243$, $n_1 \approx 49$, $n_2 \approx 5$.

10.

(a) each ball goes to the first 5 boxes out of 10 with probability $\frac{1}{2}$. Thus, the probability that all five go to the first 5 boxes is $\frac{1}{2^5} = \frac{1}{32}$

(b) this is exactly one situation out of $C(10,5)$ possible. Hence the probability is

$$\frac{1}{C(10,5)} = \frac{5!}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6} = \frac{1}{252}$$

(c) the number of ways to put 5 identical balls in 10 boxes is $C(14,5)$, the number of ways to put 5 identical balls in 5 boxes is $C(9,5)$. Since each distribution of balls have the same probability, the probability that all are in the first 5 boxes equals

$$\frac{C(9,5)}{C(14,5)} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10} = \frac{9}{132}.$$

11. Let x be the random variable which equals to the number of correct answers. It is given by binomial distribution with $p = \frac{1}{4}$, $q = \frac{3}{4}$. Then $\mu_x = 25$, $\sigma_x^2 = \frac{300}{16}$.

$$P(x > 50) = P(|x - 25| > 25) < \frac{\sigma_x^2}{25^2} = \frac{300}{10000} = 3\%.$$