SOLUTIONS FOR QUIZ # 1
MATH 114

1. If $B \subset C \subset G$ is a chain of subgroups, then

$$[G : B] = [G : C] [C : B] ,$$

because every coset gC is a disjoint union of $[C : B]$ cosets ghB. In particular,

$$[G : K \cap H] = [G : K] [K : K \cap H] .$$

Since $[G : K \cap H]$ is prime, either $[G : K] = 1$ or $[K : K \cap H] = 1$. In the former case $K = G$. In the latter case $K = K \cap H$, hence $K \subset H$. But

$$[G : K \cap H] = [G : K] = [G : H] [H : K] .$$

Again either $[G : H] = 1$ and $H = G$ or $[H : K] = 1$ and $H = K$.

2. By the first isomorphism theorem $\operatorname{Ker} F$ has order 4. Recall also that $\operatorname{Ker} F$ is a normal subgroup of S_4.

Assume that a transposition $(a, b) \in \operatorname{Ker} F$. Then any conjugate transposition also belongs to $\operatorname{Ker} F$. Therefore all transposition belong to $\operatorname{Ker} F$. Transpositions generate S_4, so $\operatorname{Ker} F = S_4$. Contradiction.

In the same way, if one 3-cycle belongs to $\operatorname{Ker} F$, then all 3-cycles belong to $\operatorname{Ker} F$. All 3-cycles generate A_4, so $A_4 \subset \operatorname{Ker} F$. Contradiction.

Finally, assume that a 4-cycle (a, b, c, d) belongs to $\operatorname{Ker} F$. Again then any 4-cycle belongs to $\operatorname{Ker} F$ because all 4-cycles are conjugate. The number of 4-cycles is 6, which is greater than the order of $\operatorname{Ker} F$. Contradiction.

There is only one possibility left. All non-identical permutations in $\operatorname{Ker} F$ are of form $(a, b) (c, d)$. Thus, $\operatorname{Ker} F$ is the Klein group.

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