PROBLEM SET # 8
MATH 114

Due March 23.

1. Let F be the splitting field of the polynomial $x^4 + 25$ over $\mathbb{Q}$. List all subfields in F and the corresponding subgroups in the Galois group.

2. Prove that the Galois group of $x^4 - 5$ is isomorphic to D_4. Hint: prove that the degree of the splitting field is 8, then recall that the Galois group is a subgroup of S_4.

3. Prove that the Galois group of $x^4 + 5x^2 + 5$ over $\mathbb{Q}$ is cyclic of order 4. Hint: use the formula for the roots.

4. Let $f(x) = x^4 + ax^2 + b \in \mathbb{Q}[x]$, $b \neq 0$.

(a) Prove that if α is a root of $f(x)$, then $-\alpha$ and $\frac{\sqrt{b}}{\alpha}$ are also roots.

(b) Prove that the degree of the splitting field is 1,2,4 or 8.

(c) Prove that the Galois group is isomorphic to $\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4$ or D_4.

5. For a cubic polynomial $f(x) = x^3 + ax + b$ the discriminant is given by the formula

$$D = -4a^3 - 27b^2.$$

Assume that a and b are real numbers. Prove that D is negative if and only if $f(x)$ has exactly one real root.

6. Assume that $f(x) = g(x)h(x)$ for some separable polynomials $f(x), g(x), h(x) \in F[x]$. Denote by E_f, E_g and E_h the splitting fields of the polynomials $f(x)$, $g(x)$ and $h(x)$ respectively. Let

$$(E_f/F) = (E_g/F)(E_h/F).$$

Prove that the Galois group of $f(x)$ is isomorphic to the direct product of the Galois groups of $g(x)$ and $h(x)$.

Date: March 16, 2006.