PROBLEM SET # 4
MATH 114

Due February 16.

Read chapter 1 in Artin's book. We assume that **all fields are commutative** in homework problems.

1. Let F be a field, and $F[i]$ denote the set of all expressions $a+bi$, with $a, b \in F$. Define addition and multiplication in $F[i]$ by

$$(a+bi)+(c+\mathrm{di})=(a+c)+(b+d)\,i,$$

$$(a+bi)(c+di)=ac-bd+(ad+bc)\,i.$$

Determine if $F[i]$ is a field for $F=\mathbb{Q}, \mathbb{R}, \mathbb{Z}_3, \mathbb{Z}_5$.

2. Assume that $\operatorname{char} F = p$. Prove that $(a+b)^p = a^p + b^p$. Hint: use binomial formula.

3. Prove the little Fermat's theorem

$$a^p \equiv a \mod p$$

for any prime p and integer a. Hint: use the previous problem.

4. Let V be a vector space of dimension n and $A : V \to V$ be a linear map such that $A^N = 0$ for some integer $N > 0$. Prove that $A^n = 0$. Hint: check that $\operatorname{Im} A^k$ is a proper subspace in $\operatorname{Im} A^{k-1}$.

5. Find a formula for a general term of the Fibonacci sequence

1,1,2,3,5,8,13,...

Hint: write the Fibonacci sequence as a linear combination of

$$1, \alpha, \alpha^2, \alpha^3, \ldots \text{ and } 1, \beta, \beta^2, \beta^3, \ldots,$$

where

$$\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}.$$

6. Let $F = \mathbb{Z}_p$.

(a) Prove that the number of one dimensional subspaces in F^n equals $\frac{p^n-1}{p-1}$;

(b) (Extra credit) Find the number of 2-dimensional subspaces in F^n.

Date: February 9, 2006.