PROBLEM SET # 2
MATH 114

Due February 2.

1. An *automorphism* of a group G is an isomorphism from G to itself. Denote by $\operatorname{Aut} G$ the set of all automorphisms of G.

(a) Prove that $\operatorname{Aut} G$ is a group with respect to the operation of composition.

(b) Let G be a finite cyclic group. Describe $\operatorname{Aut} G$.

(c) Give an example of an abelian G such that $\operatorname{Aut} G$ is not abelian.

2. Use the same notations as in Problem 1. Let π_g be the map of G to iself defined by $\pi_g(x) = gxg^{-1}$, here $g \in G$.

(a) Show that $\pi_g \in \operatorname{Aut} G$.

(b) Let $\operatorname{Inn} G = \{\pi_g \mid g \in G\}$. Show that $\operatorname{Inn} G$ is a normal subgroup in $\operatorname{Aut} G$.

3. Show that a group of order p^2 is abelian.

4. One makes necklaces from black and white beads. Let p be a prime number. Two necklaces are the same if one can be obtained from another by a rotation or a flip over. How many different necklaces of p beads one can make?

5. Assume that N is a normal subgroup of a group G. Prove that if N and G/N are solvable, then G is solvable.

6. For any permutation s denote by $F(s)$ the number of fixed points of s (k is a fixed point if $s(k) = k$). Let N be a normal subgroup of A_n. Choose a non-identical permutation $s \in N$ with maximal possible $F(s)$.

(a) Prove that any of disjoint cycles of s has length not greater than 3. (Hint: if $s \in N$, then $gsg^{-1} \in N$ for any even permutation g).

(b) Prove that the number of disjoint cycles in s is not greater than 2.

(c) Assume that $n \geq 5$. Prove that s is a 3-cycle.

(d) Use (c) to show that A_n is *simple* for $n \geq 5$, i.e. A_n does not have proper non-trivial normal subgroups. (Hint: A_n is generated by 3-cycles, as it was proven in class).

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