PROBLEM SET # 9
MATH 114

Due April 6.

1. Let $n = p$, or $2p$ where p is a prime number. Prove that the Galois group of the polynomial $x^n - 1$ over any field F is cyclic.

2. Show that the Galois group of $x^{15} - 1$ over $\mathbb{Q}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$.

By $\mathbb{F}_q$ we denote the finite field of q elements.

3. Find the Galois groups of $x^6 - 1$ over $\mathbb{F}_5$, $\mathbb{F}_{25}$ and $\mathbb{F}_{125}$.

4. Let $F \subset E$ be an extension of finite fields. Prove that

$$|E| = |F|^{(E/F)}.$$

5. Let $f(x) \in \mathbb{Z}_p[x]$ be an irreducible polynomial of degree 3. Prove that $f(x)$ is irreducible over $\mathbb{F}_{p^5}$.

6. Let $q = p^k$ for some prime p, n be a number relatively prime to p, m be the minimal positive integer such that

$$q^m \equiv 1 \mod n.$$

Show that the Galois group of $x^n - 1$ over $\mathbb{F}_q$ is isomorphic to $\mathbb{Z}_m$.

Date: March 22, 2006.