REVIEW EXERCISES
MATH 114

1. Let G be a transitive subgroup of S_n.
(a) Prove that if n is prime, then G contains an n-cycle.
(b) Show that (a) is not true if n is not prime.

2. Let F be a field such that the multiplicative group F^* is cyclic. Prove that F is finite.

3. Let G be a transitive subgroup of S_6 which contains a 5-cycle. Prove that G is not solvable.

4. Let F be a field and char $F \neq 2$, $\alpha, \beta \in F$. Prove that $F(\sqrt{\alpha}) = F\left(\sqrt{\beta}\right)$ if and only of $\alpha\beta$ is a square in F.

5. Find the minimal polynomial for

$$1 + \sqrt[3]{2} + \sqrt[3]{4}$$

over $\mathbb{Q}$.

6. Prove that any algebraically closed field is infinite.

7. Is $x^3 + x + 1$ irreducible over $\mathbb{F}_{256}$?

8. Which of the following extensions are normal

$$\mathbb{Q} \subset \mathbb{Q}\left(\sqrt{1 - \sqrt{2}}\right),$$

$$\mathbb{Q} \subset \mathbb{Q}\left(\sqrt[3]{2}, \sqrt{3}\right),$$

$$\mathbb{Q} \subset \mathbb{Q}\left(\sqrt[3]{2}, \sqrt{-3}\right)?$$

9. Determine if

$$\mathbb{Q}\left(\sqrt{1 - \sqrt{2}}\right) = \mathbb{Q}\left(\sqrt{-1}, \sqrt{2}\right).$$

10. Let $\mathbb{Q} \subset F$ be a finite normal extension such that for any two subfields E and K of F either $K \subset E$ or $E \subset K$. Then the Galois group of F over $\mathbb{Q}$ is cyclic of order p^n for some prime number p.

11. Let $F \subset B \subset E$ be a chain of extensions such that $F \subset B$ is normal and $B \subset E$ is normal. Is it always true that $F \subset E$ is normal?

12. Find the Galois group of $(x^2 - 3)(x^2 + 1)(x^3 - 6)$ over $\mathbb{Q}$.

13. Find the Galois group of $x^4 + 3x + 5$ over $\mathbb{Q}$.

14. Let p be a prime number. Prove that $\sqrt[n]{p}$ is constructible if and only if $n = 2^k$ for some k.

Date: May 6, 2006.

15. Prove that any subfield of $\mathbb{Q}\left(\sqrt[n]{2}\right)$ coincides with $\mathbb{Q}\left(\sqrt[d]{2}\right)$ for some divisor d of n.

16. Prove that there exists a polynomial of degree 7 whose Galois group over $\mathbb{Q}$ is $\mathbb{Z}_7$.

17. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of odd prime degree p solvable in radicals. Prove that the number of real roots of $f(x)$ equals p or 1.

18. Let $f(x) \in \mathbb{F}_2[x]$ be an irreducible polynomial. Prove that $f(x)$ divides $x^{256} - x$ if and only if the degree of $f(x)$ is 1,2 ,4 or 8.

19. Suppose that the Galois group over $\mathbb{Q}$ of a polynomial $f(x) \in \mathbb{Q}[x]$ has odd order. Prove that all roots of $f(x)$ are real.

20. Find the Galois group of $x^6 - 8$ over $\mathbb{Q}$.