Solutions to Quiz 2

Math 110

1. (a) True: if W is a subspace of V, then $\dim W \leq \dim V = 1$. So either $\dim W = 0$, in which case $W = \{0\}$, or $\dim W = 1 = \dim V$, in which case $W = V$.

   (b) False: for example, suppose $V = \mathbb{R}^2$, and $W = \{(x, x) : x \in \mathbb{R}\}$. Then the standard basis $\{(1,0), (0,1)\}$ of V has no vectors which are in W, but $W \neq \{0\}$, so no subset of the standard basis can be a basis of W.

2. ($\Rightarrow$) Assume T is linear; let the graph of T be called Γ_T. First, Γ_T is nonempty since $(0, T(0)) \in \Gamma_T$. Now suppose we have $(x, Tx), (y, Ty) \in \Gamma_T$; then $(x, Tx) + (y, Ty) = (x + y, Tx + Ty) = (x + y, T(x + y)) \in \Gamma_T$. Similarly, $c(x, Tx) = (cx, c(Tx)) = (cx, T(cx)) \in \Gamma_T$. Therefore, since Γ_T is nonempty and closed under addition and scalar multiplication, Γ_T is a subspace of $V \oplus W$.

   ($\Leftarrow$) Assume Γ_T is a subspace of $V \oplus W$. Suppose we have $x, y \in V$; then since $(x, Tx), (y, Ty) \in \Gamma_T$, we have $(x, Tx) + (y, Ty) = (x + y, Tx + Ty) \in \Gamma_T$ also. This implies that we must have $Tx + Ty = T(x + y)$. Similarly, $c(x, Tx) = (cx, c(Tx)) \in \Gamma_T$, which implies $c(Tx) = T(cx)$. Therefore, since T preserves addition and scalar multiplication, T is linear.