Combinatorial Differential Forms on Log Schemes

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Abstract

[Refer to Breen and Messing paper.] This article gives generalizations of the definitions given in that paper which define $\Omega^{(n)}_{X/S}$ and $\Omega^n_{X/S}$ intrinsically in the log scheme case.

Let $f : X \to S$ be a morphism of log schemes. Recall that if $\mathscr{F}$ is an $\mathscr{O}_X$-module, then a log derivation $(\mathscr{O}_X, \mathscr{M}_X) \to \mathscr{F}$ over S consists of a pair of maps $d : \mathscr{O}_X \to \mathscr{F}$ and $\delta : \mathscr{M}_X \to \mathscr{F}$ satisfying:

1. d is an $\mathscr{O}_S$-derivation.
2. If m is a section of $\mathscr{M}_X$, then $d(\alpha(m)) = \alpha(m)\delta m$, where $\alpha : \mathscr{M}_X \to \mathscr{O}_X$ gives the log structure of X.
3. δ is additive.
4. If m is a section of $\mathscr{M}_S$, then $\delta m = 0$, where m is considered to be a section of $\mathscr{M}_X$ via the map $f^\# : f^{-1}\mathscr{M}_S \to \mathscr{M}_X$ given by the morphism.

We then define $\Omega^1_{X/S}$ as the left universal sheaf with a given log derivation $(\mathscr{O}_X, \mathscr{M}_X) \to \Omega^1_{X/S}$.

We begin by giving a geometric construction of $\Omega^1_{X/S}$.

Theorem 1. *Let $P_{X/S} = \mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{O}_X$, considered as an $\mathscr{O}_X$-algebra by multiplication on the first factor, and let $Q_{X/S}$ be the $P_{X/S}$-algebra given by generators t^m for m a section of $\mathscr{M}_X$, subject to the relations*

1. $(\alpha(m) \otimes 1)t^m = 1 \otimes \alpha(m)$.
2. $t^{m+n} = t^m t^n$.
3. *If m is a section of $\mathscr{M}_S$, then $t^m = 1$.*

Let J be the ideal of $Q_{X/S}$ generated by $1 \otimes f - f \otimes 1$ and $t^m - 1$. (This can also be described as the kernel of the map $Q_{X/S} \to \mathscr{O}_X$ which extends the multiplication map $P_{X/S} \to \mathscr{O}_X$ by sending $t^m \mapsto 1$.)

Then $\Omega^1_{X/S} \simeq J/J^2$, with the log derivation given by $df = 1 \otimes f - f \otimes 1$ and $\delta m = t^m - 1$.

Proof. To check we have a log derivation $(\mathscr{O}_X, \mathscr{M}_X) \to J/J^2$, first we have that d is an $\mathscr{O}_S$-derivation by the same proof as in the non-logarithmic case. Now if $m \in \mathscr{M}_X$, then $\alpha(m)\delta m = (\alpha(m)\otimes 1)(t^m - 1) = 1 \otimes \alpha(m) - \alpha(m) \otimes 1 = d(\alpha(m))$. Also, since $(t^m - 1)(t^n - 1) \in J^2$, $\delta(m+n) = t^{m+n} - 1 = (t^m - 1) + (t^n - 1) = \delta m + \delta n$ in J/J^2. Finally, if $m \in \mathscr{M}_S$, then $\delta m = t^m - 1 = 0$.

Now suppose we have another log derivation $(d', \delta') : (\mathscr{O}_X, \mathscr{M}_X) \to \mathscr{F}$. Then any member of J can be written as $\sum_i (f_i \otimes g_i)t^{m_i}$, where $\sum_i f_i g_i = 0$. Now for each i, $(f_i \otimes g_i)t^{m_i} - f_i \otimes g_i - (f_i g_i \otimes 1)t^{m_i} + f_i g_i \otimes 1 = f_i(1 \otimes g_i - g_i \otimes 1)(t^{m_i} - 1) \in J^2$. Therefore, this sum is equivalent to $\sum_i (f_i \otimes g_i + (f_i g_i \otimes 1)t^{m_i} - f_i g_i \otimes 1) = \sum_i (f_i(1 \otimes g_i - g_i \otimes 1) + (f_i g_i \otimes 1)(t^{m_i} - 1)) = \sum_i (f_i d g_i + f_i g_i \delta m_i)$, since $\sum_i f_i g_i \otimes 1 = 0$. Thus, if the log derivation factors through J/J^2, the map $J/J^2 \to \mathscr{F}$ must be given by $\sum_i (f_i \otimes g_i)t^{m_i} \mapsto \sum_i (f_i d' g_i + f_i g_i \delta' m_i)$.

It only remains to check that the map defined in this way is well-defined, since it is obviously $\mathscr{O}_X$-linear, and it sends $df = (1 \otimes f)t^0 - (f \otimes 1)t^0 \mapsto (1\, d' f + f\, \delta' 0) - (f\, d' 1 + f\, \delta' 0) = d' f$ and $\delta m = t^m - 1 = (1 \otimes 1)t^m - (1 \otimes 1)t^0 \mapsto (1\, d' 1 + 1\, \delta' m) - (1\, d' 1 + 1\, \delta' 0) = \delta' m$. First, we check that multiples of the relations defining $Q_{X/S}$ are sent to 0. However, $Q_{X/S}$ can be expressed as a quotient of $P_{X/S}[\mathscr{M}_X]$, which already eliminates relations $t^{m+n} = t^m t^n$. Now $(f \otimes g)t^n \cdot ((\alpha(m) \otimes 1)t^m - 1 \otimes \alpha(m)) = (f\alpha(m) \otimes g)t^{m+n} - (f \otimes g\alpha(m))t^n \mapsto (f\alpha(m)\, d' g + f g \alpha(m) \delta'(m+n)) - (f\, d'(g\alpha(m)) + f g \alpha(m) \delta' n) = f\alpha(m)\, d' g - (f g\, d' \alpha(m) + f\alpha(m)\, d' g) + f g \alpha(m) \delta' m = f g (d' \alpha(m) - \alpha(m) \delta' m) = 0$.

Finally, if $m \in \mathscr{M}_S$, then $(f \otimes g)t^n \cdot (t^m - 1) = (f \otimes g)t^{m+n} - (f \otimes g)t^n \mapsto (f\, d'g + fg\delta'(m+n)) - (f\, d'g + fg\delta' n) = fg\delta' m = 0$.

Now let $(\sum_i (f_i \otimes g_i)t^{m_i})(\sum_j (a_j \otimes b_j)t^{n_j}) = \sum_{i,j}(f_i a_j \otimes g_i b_j)t^{m_i+n_j} \in J^2$, with $\sum_i f_i g_i = \sum_j a_j b_j = 0$. This element is sent to $\sum_{i,j}(f_i a_j\, d'(g_i b_j) + f_i g_i a_j b_j \delta'(m_i + n_j)) = \sum_{i,j}(f_i a_j g_i\, d' b_j + f_i a_j b_j\, d' g_i + f_i g_i a_j b_j \delta' m_i + f_i g_i a_j b_j \delta' n_j)$. Here, the sums of the first and fourth terms are zero since $\sum_i f_i g_i = 0$, and the sums of the second and third terms are zero since $\sum_j a_j b_j = 0$. Since such products generate J^2, we are done. ☐

The geometric intepretation is as follows: $\mathbf{Spec}(Q_{X/S})$ is an open subset of a log blowup of $\mathbf{Spec}(P_{X/S}) = X \times_S X$, and J is the ideal of the strict transform of the diagonal. Following [the paper], if we let $\Delta^1_{X/S} = \mathbf{Spec}(Q_{X/S}/J^2)$ be the closed subscheme defined by J^2, then $\Omega^1_{X/S}$ is just the sheaf of ideals of the diagonal in $\Delta^1_{X/S}$. Generalizing this similarly to [the paper], we give the following definition:

Definition 1. *Let $P^{(n)}_{X/S} = \bigotimes_{i=0}^n \mathscr{O}_X$, and let $p_r : \mathscr{O}_X \to P^{(n)}_{X/S}$ be the $n+1$ coprojection maps, $f \mapsto 1 \otimes \cdots 1 \otimes f \otimes 1 \cdots \otimes 1$. Let $Q^{(n)}_{X/S}$ be the $P^{(n)}_{X/S}$-algebra given by generators t^m_{rs} for $0 \le r, s \le n$, $m \in \mathscr{M}^{\mathrm{gp}}_X$, subject to the relations*

1. $p_r(\alpha(m))t^m_{rs} = p_s(\alpha(m))$.
2. $t^{m+n}_{rs} = t^m_{rs} t^n_{rs}$.
3. *If m is a section of $\mathscr{M}_S$, then $t^m_{rs} = 1$.*
4. $t^m_{qs} = t^m_{qr} t^m_{rs}$.
5. $t^m_{rr} = 1$.

Let J_{rs} be the ideal of $Q^{(n)}_{X/S}$ generated by $p_r(f) - p_s(f)$ for $f \in \mathscr{O}_X$ and $t^m_{rs} - 1$ for $m \in \mathscr{M}^{\mathrm{gp}}_X$. (This is the kernel of a natural extension of the multiplication map $P^{(n)}_{X/S} \to P^{(n-1)}_{X/S}$ which multiplies positions r and s.) Let $J^{(2)}_{0n} = \sum_{0 \le r < s \le n} J^2_{rs}$, and let $\Delta^{(n)}_{X/S} = \mathbf{Spec}(Q^{(n)}_{X/S}/J^{(2)}_{0n})$. Let $\tilde{J}_{rs}$ denote the image of J_{rs} in the ring of functions of $\Delta^{(n)}_{X/S}$. Then $\Psi^{(n)}_{X/S} = \prod_{i=1}^n \tilde{J}_{0i}$.

Note that this definition coincides with the above construction of $\Omega^1_{X/S}$ in the case $n = 1$, if we let $t^m_{01} = t^m$ and $t^m_{10} = 2 - t^m$, so that $t^m_{01} t^m_{10} \equiv 1 \pmod{J^2}$ since $(t^m - 1)^2 \in J^2$. We now need a couple lemmas before the main result on $\Psi^{(n)}_{X/S}$.

Lemma 2. $J_{qs} \subseteq J_{qr} + J_{rs}$.

Proof. We have $p_q(f) - p_s(f) = (p_q(f) - p_r(f)) + (p_r(f) - p_s(f)) \in J_{qr} + J_{rs}$. Also, $t^m_{qs} - 1 = (t^m_{qr} - 1)t^m_{rs} + (t^m_{rs} - 1) \in J_{qr} + J_{rs}$. ☐

Lemma 3. *The symmetric group S_{n+1} acting on the factors of $Q^{(n)}_{X/S}$, such that $\sigma \cdot p_r(f) = p_{\sigma(r)}(f)$ and $\sigma \cdot t^m_{rs} = t^m_{\sigma(r)\sigma(s)}$, induces the sign character on $\Psi^{(n)}_{X/S}$.*

Proof. Applying a permutation to one of the relations in $Q^{(n)}_{X/S}$ gives another relation, so the action on $Q^{(n)}_{X/S}$ is well-defined. Since this action clearly sends the ideal J_{rs} to $J_{\sigma(r)\sigma(s)}$ in $Q^{(n)}_{X/S}$, it preserves $J^{(2)}_{0n}$. Also, if $\sigma(0) = 0$ it clearly preserves $\prod_{i=1}^n \tilde{J}_{0i}$ also; otherwise, if $\sigma(0) = r$, the image of $\Psi^{(n)}_{X/S}$ is $\prod_{0 \le i \le n, i \ne r} \tilde{J}_{ri}$. But by the previous lemma, $\tilde{J}_{ri} \subseteq \tilde{J}_{r0} + \tilde{J}_{0i}$, so this is contained in $\tilde{J}_{r0} \prod_{1 \le i \le n, i \ne r}(\tilde{J}_{r0} + \tilde{J}_{0i}) = \tilde{J}_{0r} \prod_{1 \le i \le n, i \ne r} \tilde{J}_{0i} = \Psi^{(n)}_{X/S}$, since $\tilde{J}^2_{r0} = 0$ and $\tilde{J}_{0r} = \tilde{J}_{r0}$. (This last is because $p_r(f) - p_0(f) = -(p_0(f) - p_r(f))$, and $t^m_{r0} - 1 = -t^m_{r0}(t^m_{0r} - 1)$.) Therefore, we get an action on $\Psi^{(n)}_{X/S}$. Now using this symmetry, it suffices to show that transposition of 0 and 1 induces multiplication by -1. By definition, $\Psi^{(n)}_{X/S} \subseteq \tilde{J}_{01}$, so it suffices to check this on multiples of the generators of $\tilde{J}_{01}$.

However, we have $\tau_{01}p_0(f) - p_0(f) = p_1(f) - p_0(f) \in J_{01}$; $\tau_{01}p_1(f) - p_1(f) = p_0(f) - p_1(f) \in J_{01}$; and $\tau_{01}p_r(f) - p_r(f) = 0$ if $r \neq 0, 1$. Similarly, $\tau_{01}t_{01}^m - t_{01}^m = t_{10}^m - t_{01}^m = -(t_{01}^m - 1)(1 + t_{10}^m) \in J_{01}$; and $\tau_{01}t_{0r}^m - t_{0r}^m = t_{1r}^m - t_{0r}^m = -(t_{01}^m - 1)t_{1r}^m \in J_{01}$ for $r > 1$. Therefore, for every $x \in Q_{X/S}^{(n)}$, $\tau_{01}x \equiv x \pmod{J_{01}}$, since τ_{01} respects the ring structure, and the ring is generated by elements $p_r(f)$ and t_{0r}^m.

In addition, $\tau_{01}(p_0(f) - p_1(f)) + (p_0(f) - p_1(f)) = 0$, and $\tau_{01}(t_{01}^m - 1) + (t_{01}^m - 1) = t_{01}^m + t_{10}^m - 2 = t_{10}^m(t_{01}^m - 1)^2 \in \tilde{J}_{01}^2 = 0$. Therefore, if y is one of these generators, $\tau_{01}(xy) + xy = \tau_{01}x \cdot \tau_{01}y + xy = (\tau_{01}x - x)\tau_{01}y + x(\tau_{01}y + y) \in \tilde{J}_{01}^2 = 0$ since $\tau_{01}y \in \tilde{J}_{10} = \tilde{J}_{01}$. □

Theorem 4. *There is a natural isomorphism $\Omega_{X/S}^{(n)} \simeq \Psi_{X/S}^{(n)}$, where $\Omega_{X/S}^{(n)} = \bigwedge^{(n)} \Omega_{X/S}^1$ is the nth antisymmetric product of $\Omega_{X/S}^1$.*

Proof. The isomorphisms are $\lambda : \Omega_{X/S}^{(n)} \to \Psi_{X/S}^{(n)}$ induced by the map $\bigotimes_{\mathscr{O}_X}^n Q_{X/S} \to Q_{X/S}^{(n)}$ given by $(f_1 \otimes g_1)t^{m_1} \otimes \cdots \otimes (f_n \otimes g_n)t^{m_n} \mapsto (f_1 \cdots f_n \otimes g_1 \otimes \cdots \otimes g_n)t_{01}^{m_1} \cdots t_{0n}^{m_n}$, and $\mu : \Psi_{X/S}^{(n)} \to \Omega_{X/S}^{(n)}$ induced by the map $Q_{X/S}^{(n)} \to \Omega_{X/S}^{(n)}$ given by $(f_0 \otimes f_1 \otimes \cdots \otimes f_n)t_{01}^{m_1} \cdots t_{0n}^{m_n} \mapsto f_0(df_1 + f_1\delta m_1)\,\tilde{\wedge} \cdots \tilde{\wedge}(df_n + f_n\delta m_n)$. (Here we write $t_{rs}^m = t_{r0}^m t_{0s}^m = t_{0r}^{-m}t_{0s}^m$, and we extend $Q_{X/S}$ to contain t^m for $m \in \mathscr{M}_X^{\mathrm{gp}}$, which doesn't change the calculation of $\Omega_{X/S}^1$ since as we saw above t^m is a unit in $Q_{X/S}/J^2$ for $m \in \mathscr{M}_X$.)

To check that λ is well-defined, first observe that the image under p_r of any relation in $Q_{X/S}$ gets sent to a corresponding relation in $Q_{X/S}^{(n)}$ with t^m replaced by t_{0r}^m, so the map $\bigotimes^n Q_{X/S} \to Q_{X/S}^{(n)}$ is well-defined. Now the image of p_rJ is J_{0r}, so the image of $J \otimes \cdots \otimes J^2 \otimes \cdots \otimes J$ is contained in $J_{0r}^2 \subseteq J_{0n}^{(2)}$, and the image of $\bigotimes^n J = \prod_{i=1}^n p_iJ$ is $\prod_{i=1}^n J_{0i}$. Thus, we have a well-defined map $\bigotimes_{\mathscr{O}_X}^n \Omega_{X/S}^1 \to \Psi_{X/S}^{(n)}$. Under this map, permuting the factors of $\bigotimes^n Q_{X/S}$ corresponds to permuting the corresponding factors in $Q_{X/S}^{(n)}$ (leaving factor 0 fixed), so by the lemma on permutations of $\Psi_{X/S}^{(n)}$, it induces a map $\lambda : \Omega_{X/S}^{(n)} \to \Psi_{X/S}^{(n)}$.

To check that μ is well-defined, we first check that multiples of relations in $Q_{X/S}^{(n)}$ get sent to zero. However, multiples of relations $t_{rs}^{m+n} - t_{rs}^m t_{rs}^n$, $t_{qs}^m - t_{qr}^m t_{rs}^m$, and $t_{rr}^m - 1$ are already eliminated by expressing $Q_{X/S}^{(n)}$ as a quotient of $P_{X/S}^{(n)}[\mathscr{M}_X^{\mathrm{gp}}, \ldots, \mathscr{M}_X^{\mathrm{gp}}]$ as above. The check that multiples of $p_0(\alpha(m))t_{0r}^m - p_r(\alpha(m))$ and $t_{0r}^m - 1$ for $m \in \mathscr{M}_S$ get sent to zero is similar to the check in the proof that $\Omega_{X/S}^1 \simeq J/J^2$. Now $p_r(\alpha(m))t_{rs}^m - p_s(\alpha(m)) = [p_0(\alpha(m))t_{0s}^m - p_s(\alpha(m))] - t_{rs}^m[p_0(\alpha(m))t_{0r}^m - p_r(\alpha(m))]$, and $t_{rs}^m - 1 = t_{r0}^m(t_{0s}^m - 1) - t_{r0}^m(t_{0r}^m - 1)$, so checking these cases is sufficient.

Again, checking that J_{0r}^2 gets mapped to zero is similar to the check that J^2 is mapped to zero in the proof that $\Omega_{X/S}^1 \simeq J/J^2$. Now for the rest of $J_{0n}^{(2)}$, it suffices by symmetry to check that the image of J_{12}^2 is zero. Thus, if we denote $d^{r,s}f = p_s(f) - p_r(f)$, we have $d^{1,2}fd^{1,2}g = (d^{0,2}f - d^{0,1}f)(d^{0,2}g - d^{0,1}g) \equiv -d^{0,1}fd^{0,2}g - d^{0,1}gd^{0,2}f \pmod{J_{01}^2 + J_{02}^2}$. Now if we multiply by $(f_0 \otimes \cdots \otimes f_n)t_{01}^{m_1} \cdots t_{0n}^{m_n}$, then since $p_1(f_1)t_{01}^{m_1} - p_0(f_1) \in J_{01}$ and $p_2(f_2)t_{02}^{m_2} - p_0(f_2) \in J_{02}$, while $d^{1,2}fd^{1,2}g \in J_{01}^2 + J_{02}^2 + J_{01}J_{02}$, it is equivalent $\pmod{J_{01}^2 + J_{02}^2}$ to multiply by $(f_0f_1f_2 \otimes 1 \otimes 1 \otimes f_3 \otimes \cdots \otimes f_n)t_{03}^{m_3} \cdots t_{0n}^{m_n}$. Now this product gets sent to $f_0f_1f_2(-df\,\tilde{\wedge}\,dg - dg\,\tilde{\wedge}\,df)\,\tilde{\wedge}(df_3 + f_3\delta m_3)\,\tilde{\wedge} \cdots \tilde{\wedge}(df_n + f_n\delta m_n) = 0$. If we let $\delta^{r,s}m = t_{rs}^m - 1$, then we get similar proofs for $\delta^{1,2}md^{1,2}f$ and $\delta^{1,2}m\delta^{1,2}m'$, using the identity $\delta^{1,2}m = t_{10}^m\delta^{0,2}m - t_{10}^m\delta^{0,1}m$.

Now to see $\mu\lambda = 1$, it suffices to check that this relation holds on generators $df_1\,\tilde{\wedge} \cdots \tilde{\wedge}\,df_i\,\tilde{\wedge}\,\delta m_{i+1}\,\tilde{\wedge} \cdots \tilde{\wedge}\,\delta m_n$ of $\Omega_{X/S}^{(n)}$. This generator gets sent by λ to $d^{0,1}f_1 \cdots d^{0,i}f_i\delta^{0,i+1}m_{i+1} \cdots \delta^{0,n}m_n$. Now $p_0(f_1)d^{0,2}f_2 \cdots \delta^{0,n}m_n$ has 1 in position 1 and has a power t_{01}^0, so this is in the kernel of μ. Similarly, by doing the same with the other factors, this image is equivalent to $p_1(f_1) \cdots p_i(f_i)t_{0,i+1}^{m_{i+1}} \cdots t_{0n}^{m_n}$, which gets sent by μ to $df_1\,\tilde{\wedge} \cdots \tilde{\wedge}\,df_i\,\tilde{\wedge}\,\delta m_{i+1}\,\tilde{\wedge} \cdots \tilde{\wedge}\,\delta m_n$.

Finally, to see $\lambda\mu = 1$, we claim that on $Q_{X/S}^{(n)}$, $\lambda\mu = \prod_{i=1}^n(1 - M_i)$, where $M_i : Q_{X/S}^{(n)} \to Q_{X/S}^{(n)}/J_{0n}^{(2)}$ is defined by $(f_0 \otimes \cdots \otimes f_n)t_{01}^{m_1} \cdots t_{0n}^{m_n} \mapsto (f_0f_i \otimes \cdots \otimes 1 \otimes \cdots \otimes f_n)t_{01}^{m_1} \cdots t_{0i}^0 \cdots t_{0n}^{m_n}$. To see this, we have $df_i + f_i\delta m_i = 1 \otimes f_i - f_i \otimes 1 + f_i(t^{m_i} - 1 \otimes 1)$. However, since $(1 \otimes f_i - f_i \otimes 1)(t^{m_i} - 1) \in J^2$, this is equivalent to $(1 \otimes f_i)t^{m_i} - f_i \otimes 1$. The claim is now straightforward to check.

It is also easy to see that $M_i(J_{0i}) = 0$; $M_i(J_{ir}) = J_{0r}$ for $r \neq 0$; and $M_i(J_{rs}) = J_{rs}$ for $r, s \neq i$. Therefore, M_i induces a map $Q_{X/S}^{(n)}/J_{0n}^{(2)} \to Q_{X/S}^{(n)}/J_{0n}^{(2)}$, and M_i is zero on $\tilde{J}_{0i} \supseteq \Omega_{X/S}^{(n)}$. Therefore, since the maps M_i clearly commute, $\lambda\mu = 1$ on $\Omega_{X/S}^{(n)}$. □

(Note: this proof also shows that $\bigcap_{i=1}^n \tilde{J}_{0i} = \prod_{i=1}^n \tilde{J}_{0i} = \Omega_{X/S}^{(n)}$.)