Homework 8 Solutions

Math 110

Section 5.1

3. (a) The characteristic polynomial of A is $(1-t)(2-t)-6 = t^2-3t-4 = (t-4)(t+1)$; therefore, the eigenvalues of A are 4 and -1.
For $\lambda = 4$, we have
$$E_4 = NS\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} = \text{span}\{(2,3)\}.$$
Similarly, for $\lambda = -1$, we have
$$E_1 = NS\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} = \text{span}\{(-1,1)\}.$$
Thus, $\{(2,3),(-1,1)\}$ is a basis for $\mathbb{R}^2$ consisting of eigenvectors; we thus get $Q^{-1}AQ = D$ for
$$Q = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}, D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}.$$

4. (h) Let $\varepsilon = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ be the standard basis for $M_{2\times 2}(\mathbb{R})$. Then we see $TE_{11} = E_{22}$, $TE_{12} = E_{12}$, $TE_{21} = E_{21}$, and $TE_{22} = E_{11}$. Thus,
$$[T]_\varepsilon = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$
Now the characteristic polynomial of T is
$$\det\begin{bmatrix} -t & 0 & 0 & 1 \\ 0 & 1-t & 0 & 0 \\ 0 & 0 & 1-t & 0 \\ 1 & 0 & 0 & -t \end{bmatrix}.$$
Expanding by minors along the second and third columns, this is equal to $(1-t)^2[(-t)^2 - 1^2] = (t-1)^2(t^2-1) = (t-1)^3(t+1)$. Therefore, the eigenvalues of T are 1 and -1.
Now the eigenspace of 1 corresponds to
$$NS\begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} = \text{span}\{(0,1,0,0),(0,0,1,0),(1,0,0,1)\}.$$
In $M_{2\times 2}(\mathbb{R})$, this corresponds to $\text{span}\{E_{12}, E_{21}, E_{11}+E_{22}\}$. On the other hand, the eigenspace of -1 corresponds to
$$NS\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \text{span}\{(-1,0,0,1)\}.$$

In $M_{2\times 2}(\mathbb{R})$, this corresponds to $\operatorname{span}\{-E_{11}+E_{22}\}$.
Therefore,

$$\beta = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

diagonalizes T.

8. (a) We see that zero is an eigenvalue of T if and only if $Tx = 0$ for some nonzero vector $x \in V$, or in other words if and only if T is not one-to-one. Thus, T is one-to-one if and only if zero is not an eigenvalue of T; however, since V is finite-dimensional, T is invertible if and only if T is one-to-one.

   (b) ($\Rightarrow$) Suppose λ is an eigenvalue of T, and find a corresponding eigenvector $x \in V$. Then since $Tx = \lambda x$, we also have $T(\lambda^{-1}x) = x$, so $T^{-1}x = \lambda^{-1}x$. Therefore, λ^{-1} is an eigenvalue of T^{-1}.

   ($\Leftarrow$) If λ^{-1} is an eigenvalue of T^{-1}, then by the previous part, $\lambda = (\lambda^{-1})^{-1}$ is an eigenvalue of $T = (T^{-1})^{-1}$.

   (c) If $A \in M_{n\times n}(F)$, then A is invertible if and only if 0 is not an eigenvalue of A. Also, if A is invertible, then λ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of A^{-1}. (These results follow from the previous parts by letting $T = L_A$.)

14. Since $(A - tI)^t = A^t - tI$, we have

$$\det(A - tI) = \det[(A - tI)^t] = \det(A^t - tI).$$

15. (a) We prove this by induction on m; for $m = 1$, the statement is exactly what we are assuming. Now suppose x is an eigenvector of T^m with eigenvalue λ^m; then we calculate

$$T^{m+1}x = T(T^m x) = T(\lambda^m x) = \lambda^m(Tx) = \lambda^m(\lambda x) = \lambda^{m+1}x.$$

   Therefore, x is also an eigenvector of T^{m+1} with eigenvalue λ^{m+1}, completing the induction.

   (b) Let $A \in M_{n\times n}(F)$, and let $x \in F^n$ be an eigenvector of A with eigenvalue λ. Then for any positive integer m, x is an eigenvector of A^m with eigenvalue λ^m. (This follows from the previous part by letting $T = L_A$.)

21. (a) The ij-entry of $A - tI$ is $A_{ij} - t\delta_{ij}$. Therefore, we get

$$f(t) = \det(A - tI) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)(A_{\sigma(1),1} - t\delta_{\sigma(1),1})(A_{\sigma(2),2} - t\delta_{\sigma(2),2})\cdots(A_{\sigma(n),n} - t\delta_{\sigma(n),n}).$$

   If σ is the identity permutation, this term gives exactly $(A_{11} - t)(A_{22} - t)\cdots(A_{nn} - t)$. On the other hand, if σ is any other permutation, then $\sigma(i) \neq i$ for at least *two* values of i. (This is because if $\sigma(i) = j \neq i$ for some i, then $\sigma(j)$ also cannot be equal to j.) Therefore, any other term of the sum gives a polynomial of degree at most $n - 2$, so the sum of all the other terms is also a polynomial $q(t)$ of degree at most $n - 2$.

   (b) From the previous part, the coefficient of t^{n-1} in $f(t)$ is equal to the coefficient of t^{n-1} in $(A_{11} - t)(A_{22} - t)\cdots(A_{nn} - t)$, which is $(-1)^{n-1}(A_{11} + A_{22} + \cdots + A_{nn})$. Therefore, $a_{n-1} = (-1)^{n-1}(\operatorname{tr} A)$, so $\operatorname{tr} A = (-1)^{n-1}a_{n-1}$.

X1. If A is the given matrix, then $A(x_1, x_2, \ldots, x_n) = (x_2, \ldots, x_n, -a_0x_1 - a_1x_2 - \cdots - a_{n-1}x_n)$. Thus, if $x = (x_1, \ldots, x_n)$ is an eigenvector of A with eigenvalue λ, we must have $x_2 = \lambda x_1$, $x_3 = \lambda x_2 = \lambda^2 x_1$, $x_4 = \lambda x_3 = \lambda^3 x_1$, ..., $x_n = \lambda^{n-1}x_1$. Now plugging in, the last entry of Ax is $x_1(-a_0 - a_1\lambda - \cdots - a_{n-1}\lambda^{n-1}) = x_1\lambda^n$. Therefore, for any nonzero scalar c, $c(1, \lambda, \lambda^2, \ldots, \lambda^{n-1})$ is an eigenvector of A (and in fact, every eigenvector corresponding to λ is of this form).

X2. By the calculations in the previous problem, the eigenspace of each eigenvalue λ has dimension 1. Therefore, A is diagonalizable if and only if the characteristic polynomial $(-1)^n(t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0)$ splits, and each eigenvalue has multiplicity 1. This is equivalent to the condition that the polynomial $t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0$ have n distinct roots.

Section 5.2

7. If we apply the diagonalization process to A, we find that A has eigenvalues 5 and -1, with corresponding eigenvectors $(1, 1)$ and $(-2, 1)$. Therefore, $A = QDQ^{-1}$, where

$$D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}, Q = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}, Q^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}.$$

We thus get

$$A^n = QD^nQ^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5^n & 0 \\ 0 & (-1)^n \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 5^n + 2(-1)^n & 2[5^n - (-1)^n] \\ 5^n - (-1)^n & 2 \cdot 5^n + (-1)^n \end{bmatrix}.$$

8. From the given assumptions, we see that the multiplicity of λ_1 must be at least $n-1$, and the multiplicity of λ_2 must be at least 1. However, the sum of these two multiplicities is also at most n. Therefore, the multiplicity of λ_1 is exactly $n-1$, and the multiplicity of λ_2 is exactly 1.

   This implies that λ_1 and λ_2 are the only eigenvalues; also, $(\lambda_1 - t)^{n-1}(\lambda_2 - t)$ must divide the characteristic polynomial. But $(\lambda_1 - t)^{n-1}(\lambda_2 - t)$ and the characteristic polynomial both have leading term $(-1)^n t$, so in fact they are equal. Thus, the characteristic polynomial splits; $\dim(E_{\lambda_1}) = n - 1$ is equal to the multiplicity of λ_1; and $1 \le \dim(E_{\lambda_2}) \le 1$ since λ_2 has multiplicity 1, so $\dim(E_{\lambda_2}) = 1$ is equal to the multiplicity of λ_2. Therefore, A is diagonalizable.

   Alternately, taking a basis of E_{λ_1} and adjoining an eigenvector corresponding to λ_2 gives a linearly independent subset of V by theorem 5.8 from the book; this set has n vectors, so it must be a basis.

11. Since A is similar to an upper triangular matrix, the characteristic polynomial of A splits. Therefore, it must be equal to $f(t) = (\lambda_1 - t)^{m_1}(\lambda_2 - t)^{m_2} \cdots (\lambda_k - t)^{m_k}$. Thus, $\det A = \det(A - 0 \cdot I) = f(0) = \lambda_1^{m_1}\lambda_2^{m_2} \cdots \lambda_k^{m_k}$. Also, the coefficient of t^{n-1} in this polynomial is $(-1)^{n-1}(m_1\lambda_1 + m_2\lambda_2 + \cdots + m_k\lambda_k)$, so by problem 5.1.21, the trace of A is equal to $m_1\lambda_1 + m_2\lambda_2 + \cdots + m_k\lambda_k$.

21. We use induction on k; the case $k = 1$ is trivial, and the case $k = 2$ follows directly from problem 1.6.33 from a previous homework. Now for the general inductive step, let $W_1 = \operatorname{span}(\beta_1 \cup \cdots \beta_{k-1})$, $W_2 = \operatorname{span}(\beta_k)$. Then by induction, $W_1 = \operatorname{span}(\beta_1) \oplus \cdots \oplus \operatorname{span}(\beta_{k-1})$, and $V = W_1 \oplus W_2$. We claim that this implies $V = \operatorname{span}(\beta_1) \oplus \cdots \oplus \operatorname{span}(\beta_{k-1}) \oplus \operatorname{span}(\beta_k)$. To see this, first any $x \in V$ can be written $y + z$ with $y \in W_1$, $z \in W_2$. Also, we can write $y = y_1 + \cdots + y_{k-1}$

with $y_1 \in \text{span}(\beta_1)$, $\ldots$, $y_{k-1} \in \text{span}(\beta_{k-1})$. This, $x = y_1 + \cdots + y_{k-1} + z \in \text{span}(\beta_1) + \cdots + \text{span}(\beta_{k-1}) + \text{span}(\beta_k)$.

Also, suppose that $y_1 + \cdots + y_{k-1} + y_k = 0$ with $y_1 \in \text{span}(\beta_1)$, $\ldots$, $y_k \in \text{span}(\beta_k)$. Then $y_k = -y_1 - \cdots - y_{k-1} \in W_1 \cap W_2$, so $y_k = 0$. This implies that $y_1 + \cdots + y_{k-1} = 0$, so $y_1 = \cdots = y_{k-1} = 0$ also since $W_1 = \text{span}(\beta_1) \oplus \cdots \oplus \text{span}(\beta_{k-1})$.

X1. (a) Let x_k denote the kth vector; we then see that $Ax_k = (\omega^k + \omega^{6k}, 1 + \omega^{2k}, \omega^k + \omega^{3k}, \ldots, \omega^{4k} + \omega^{6k}, 1 + \omega^{5k})$. However, since $\omega^7 = 0$, we have $\omega^{6k} = \omega^{-k}$ and $1 = \omega^{7k}$, so $Ax_k = (\omega^k + \omega^{-k})x_k$. Therefore, x_k is an eigenvector of A, with eigenvalue $\omega^k + \omega^{-k}$. However, since $\omega^k = \cos(\frac{2\pi k}{7}) + i\sin(\frac{2\pi k}{7})$ and $\omega^{-k} = \cos(\frac{2\pi k}{7}) - i\sin(\frac{2\pi k}{7})$, this eigenvalue is equal to $2\cos(\frac{2\pi k}{7})$.

(b) For $k = 1, 2, 3$, the eigenspace of $2\cos(\frac{2\pi k}{7})$ has two linearly independent vectors x_k, x_{-k}. Therefore, $\{x_{-3}, x_{-2}, \ldots, x_3\}$ is linearly independent by 5.8, so it is a basis of $\mathbb{C}^7$ consisting of eigenvectors of A.

(In fact, A is diagonalizable over $\mathbb{R}$ as well: all the eigenvalues are real, and for $k = 1, 2, 3$, $\frac{1}{2}(x_k + x_{-k}) = (1, \cos(\frac{2\pi k}{7}), \ldots, \cos(\frac{12\pi k}{7}))$ and $\frac{1}{2i}(x_k - x_{-k}) = (0, \sin(\frac{2\pi k}{7}), \ldots, \sin(\frac{12\pi k}{7}))$ are linearly independent real eigenvectors corresponding to $2\cos(\frac{2\pi k}{7})$. Since $x_0 = (1, 1, \ldots, 1)$, we thus get a basis of $\mathbb{R}^7$ consisting of real eigenvectors of A.)

Alternately, A is symmetric, and we will see later that any symmetric matrix is diagonalizable.

(c) If we move the top row of A to the bottom, this does not change the sign of the determinant, while the result is almost upper triangular. We now reduce A as follows:

$$\begin{bmatrix} 1&0&1&0&0&0&0\\ 0&1&0&1&0&0&0\\ 0&0&1&0&1&0&0\\ 0&0&0&1&0&1&0\\ 0&0&0&0&1&0&1\\ 1&0&0&0&0&1&0\\ 0&1&0&0&0&0&1 \end{bmatrix} \to \begin{bmatrix} 1&0&1&0&0&0&0\\ 0&1&0&1&0&0&0\\ 0&0&1&0&1&0&0\\ 0&0&0&1&0&1&0\\ 0&0&0&0&1&0&1\\ 0&0&-1&0&0&1&0\\ 0&0&0&-1&0&0&1 \end{bmatrix} \to \begin{bmatrix} 1&0&1&0&0&0&0\\ 0&1&0&1&0&0&0\\ 0&0&1&0&1&0&0\\ 0&0&0&1&0&1&0\\ 0&0&0&0&1&0&1\\ 0&0&0&0&1&1&0\\ 0&0&0&0&0&1&1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1&0&1&0&0&0&0\\ 0&1&0&1&0&0&0\\ 0&0&1&0&1&0&0\\ 0&0&0&1&0&1&0\\ 0&0&0&0&1&0&1\\ 0&0&0&0&0&1&-1\\ 0&0&0&0&0&1&1 \end{bmatrix} \to \begin{bmatrix} 1&0&1&0&0&0&0\\ 0&1&0&1&0&0&0\\ 0&0&1&0&1&0&0\\ 0&0&0&1&0&1&0\\ 0&0&0&0&1&0&1\\ 0&0&0&0&0&1&-1\\ 0&0&0&0&0&0&2 \end{bmatrix}.$$

None of these steps changes the determinant; thus, $\det A = 2$.

(d) From part (b), the eigenvectors of A are 2 with multiplicity 1 and $2\cos(\frac{2\pi}{7})$, $2\cos(\frac{4\pi}{7})$, $2\cos(\frac{6\pi}{7})$ each with multiplicity 2. Since the product of the eigenvectors is equal to $\det A$, we thus get

$$2^7 \cos^2\left(\frac{2\pi}{7}\right) \cos^2\left(\frac{4\pi}{7}\right) \cos^2\left(\frac{6\pi}{7}\right) = 2.$$

Therefore, $\cos(\frac{2\pi}{7})\cos(\frac{4\pi}{7})\cos(\frac{6\pi}{7}) = \pm\frac{1}{8}$. However, $\cos(\frac{2\pi}{7})$ is positive, while $\cos(\frac{4\pi}{7})$ and $\cos(\frac{6\pi}{7})$ are negative, so we conclude

$$\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{6\pi}{7}\right) = \frac{1}{8}.$$

Section 5.4

2. (a) If $f \in W$, then f has degree at most 2, so f' has degree at most $1 \leq 2$, which implies $f' \in W$ also. Therefore, W is T-invariant.

   (e) In this case, W is not T-invariant; for example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in W$, but $T(A) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \notin W$.

6. (a) We calculate $T(e_1) = (1,0,1,1)$; $T^2(e_1) = (1,-1,2,2)$; and $T^3(e_1) = (0,-3,3,3) = 3T^2(e_1) - 6T(e_1) + 3e_1$. Therefore, the T-cyclic subspace generated by e_1 is $\text{span}\{(1,0,0,0), (1,0,1,1), (1,-1,2,2)\}$. This subspace can also be identified as $\{(a,b,c,d) : c = d\}$.

17. We can rewrite any linear combination $c_0 I_n + c_1 A + c_2 A^2 + \cdots + c_k A^k$ as $p(A)$, where $p(t) = c_k t^k + \cdots + c_1 t + c_0$. We now divide $p(t)$ by the characteristic polynomial $f(t)$ of A to get $p(t) = q(t)f(t) + r(t)$, where the degree of $r(t)$ is less than n. This implies that $p(A) = q(A)f(A) + r(A)$; however, by the Cayley-Hamilton theorem, $f(A) = 0$. Therefore, $p(A) = r(A)$ is a linear combination of $I_n, A, A^2, \ldots, A^{n-1}$.

    We have thus shown that $\text{span}\{I_n, A, A^2, \ldots\} \subseteq \text{span}\{I_n, A, A^2, \ldots, A^{n-1}\}$; since the right hand side clearly has dimension at most n, this implies the desired result.

    Alternately, consider the linear transformation $T : M_{n\times n}(F) \to M_{n\times n}(F)$ such that $T(B) = AB$. Then the T-invariant subspace generated by I_n is exactly $\text{span}\{I_n, A, A^2, \ldots\}$; but since the Cayley-Hamilton therom implies that A^n is a linear combination of $I_n, A, A^2, \ldots, A^{n-1}$, our general results on T-cyclic subspaces imply that $\text{span}\{I_n, A, A^2, \ldots\}$ has dimension at most n.

42. Since A has rank 1, the null space of A has dimension $n-1$, which implies that A has eigenvalue 0 with multiplicity at least $n-1$. On the other hand, A also has eigenvalue n with corresponding eigenvector $x = (1,1,\ldots,1)$. Therefore, the characteristic polynomial of A must be $(-t)^{n-1}(n-t)$.