Homework 7 Solutions

Math 110

Section 3.4

2. (j) Reducing the augmented matrix

$$\begin{bmatrix} 2 & 0 & 3 & 0 & -4 & 5 \\ 3 & -4 & 8 & 3 & 0 & 8 \\ 1 & -1 & 2 & 1 & -1 & 2 \\ -2 & 5 & -9 & -3 & -5 & -8 \end{bmatrix}$$

to reduced row echelon form results in

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 & -1 \end{bmatrix}.$$

Therefore, the general solution to the given system of equations is $(x_1, x_2, x_3, x_4, x_5) = (2x_5 + 1, 3x_5, 1, 2x_5 - 1, x_5) = (1, 0, 1, -1, 0) + x_5(2, 3, 0, 2, 1)$.

5. Letting a_j denote the jth column of A, from the given reduced row echelon form we see that

$$a_3 = 2a_1 - 5a_2 = 2\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} - 5\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix},$$

and

$$a_5 = -2a_1 - 3a_2 + 6a_4 = -2\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} - 3\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + 6\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ -9 \end{bmatrix}.$$

Therefore,

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 & 4 \\ -1 & -1 & 3 & -2 & -7 \\ 3 & 1 & 1 & 0 & -9 \end{bmatrix}.$$

Section 4.1

3. (a) $\det \begin{bmatrix} -1+i & 1-4i \\ 3+2i & 2-3i \end{bmatrix} = (-1+i)(2-3i) - (3+2i)(1-4i) = (1+5i) - (11-10i) = -10+15i.$

(b) $\det \begin{bmatrix} 5-2i & 6+4i \\ -3+i & 7i \end{bmatrix} = (5-2i)(7i) - (-3+i)(6+4i) = (14+35i) - (-22-6i) = 36+41i.$

(c) $\det \begin{bmatrix} 2i & 3 \\ 4 & 6i \end{bmatrix} = (2i)(6i) - 3 \cdot 4 = -12 - 12 = -24.$

7. Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$; then $\det A = ad - bc$, while

$$\det A^t = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - cb = \det A.$$

Section 4.2

4. We have

$$\begin{bmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

Therefore,

$$\det \begin{bmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{bmatrix} = \det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = 2 \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

In other words, $k = 2$.

14. Switching the first and third rows gives a lower triangular matrix; thus,

$$\det \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 0 \\ 7 & 0 & 0 \end{bmatrix} = -\det \begin{bmatrix} 7 & 0 & 0 \\ 5 & 6 & 0 \\ 2 & 3 & 4 \end{bmatrix} = -(7 \cdot 6 \cdot 4) = -168.$$

18. Using Gaussian elimination on the matrix, we get

$$\det \begin{bmatrix} 1 & -2 & 3 \\ -1 & 2 & -5 \\ 3 & -1 & 2 \end{bmatrix} = \det \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & -2 \\ 0 & 5 & -7 \end{bmatrix} = -\det \begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -7 \\ 0 & 0 & -2 \end{bmatrix} = -(1 \cdot 5 \cdot (-2)) = 10.$$

25. We can get kA from A by multiplying each column by k one at a time; therefore, we multiply the determinant by k n times, so $\det(kA) = k^n \det A$.

30. We get B by permuting the rows of A according to the permutation

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ n & n-1 & n-2 & \cdots & 1 \end{pmatrix}.$$

This permutation has $\frac{n(n-1)}{2}$ inversions, so it has sign $(-1)^{n(n-1)/2}$. Therefore, $\det B = (-1)^{n(n-1)/2} \det A$.

Alternately, we can get B from A by switching rows 1 and n, then rows 2 and $n-1$, and so on up to switching rows $\lfloor \frac{n}{2} \rfloor$ and $n + 1 - \lfloor \frac{n}{2} \rfloor$. (Here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.) This involves $\lfloor \frac{n}{2} \rfloor$ switches, so $\det B = (-1)^{\lfloor n/2 \rfloor} \det A$.

Note that both of these actually give the same sign: they both give $\det B = \det A$ if $n \equiv 0$ or $1 \pmod 4$, and $\det B = -\det A$ if $n \equiv 2$ or $3 \pmod 4$.

Section 4.3

12. Since $\det(Q^t) = \det Q$, and $(\det Q)(\det Q^t) = \det(QQ^t) = \det I = 1$, we get $(\det Q)^2 = 1$, so $\det Q = \pm 1$.

15. If A and B are similar, then $B = QAQ^{-1}$ for some invertible matrix Q. Therefore,

$$\det B = (\det Q)(\det A)(\det Q^{-1}) = (\det Q)(\det A) \cdot \frac{1}{\det Q} = \det A.$$

21. Suppose A is $k \times k$ and C is $(n-k) \times (n-k)$. Then we use the formula

$$\det M = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n}.$$

In order for one of these terms to be nonzero, we must have $\sigma(i) \le k$ for $i = 1, \ldots, k$; therefore, σ decomposes into a permutation τ of $\{1, \ldots, k\}$ and a permutation τ' of $\{k+1, \ldots, n\}$. It is straightforward to check that $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\tau)\operatorname{sgn}(\tau')$, since any inversion in σ comes from an inversion in τ or an inversion in τ'. Therefore,

$$\begin{aligned}
\det M &= \sum_{\tau \in S_k} \sum_{\tau' \in S_{n-k}} \operatorname{sgn}(\tau)\operatorname{sgn}(\tau') a_{\tau(1),1} \cdots a_{\tau(k),k} a_{k+\tau'(1),k+1} \cdots a_{k+\tau'(n-k),n} \\
&= \left(\sum_{\tau \in S_k} \operatorname{sgn}(\tau) a_{\tau(1),1} \cdots a_{\tau(k),k} \right) \left(\sum_{\tau' \in S_{n-k}} \operatorname{sgn}(\tau') a_{k+\tau'(1),k+1} \cdots a_{k+\tau'(n-k),n} \right) \\
&= (\det A)(\det C).
\end{aligned}$$

Alternately, one could use induction on k and prove the inductive step by expanding by minors along the first column.

Yet another solution is: first, it is easy to prove the desired statement either if $A = I_k$ or if $C = I_{n-k}$. Now in general, if $\det A = 0$, then the columns of A are linearly dependent, so the first k columns of M are linearly dependent, and thus $\det M = 0$. Otherwise, A is invertible, and we can write

$$M = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & C \end{bmatrix}.$$

Therefore,

$$\det M = \det \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \det \begin{bmatrix} I & A^{-1}B \\ 0 & C \end{bmatrix} = (\det A)(\det C).$$

23. (a) By hypothesis, we can find a $k \times k$ submatrix with nonzero determinant. Since this submatrix has rank k, its columns are linearly independent, which implies that the corresponding columns of A are also linearly independent. Therefore, $\operatorname{rank}(A) \ge k$. Once we prove (b), that will show that $\operatorname{rank}(A) \le k$; otherwise, if $\operatorname{rank}(A)$ were some number $m > k$, we would be able to find an $m \times m$ submatrix with nonzero determinant, contradicting the construction of k.

    (b) Since $\operatorname{rank}(A) = k$, we can find k linearly independent columns of A; selecting these columns of A gives an $m \times k$ submatrix which still has rank k. Therefore, we can also find k linearly independent rows of this submatrix; selecting these rows gives a $k \times k$ submatrix which has rank k and therefore has nonzero determinant.

24. For convenience, we will do the case $n = 5$; it should be clear from this how to do the general case. First, we have

$$A + tI = \begin{bmatrix} t & 0 & 0 & 0 & a_0 \\ -1 & t & 0 & 0 & a_1 \\ 0 & -1 & t & 0 & a_2 \\ 0 & 0 & -1 & t & a_3 \\ 0 & 0 & 0 & -1 & t + a_4 \end{bmatrix}.$$

Adding t times the second row to the first row gives

$$\det(A+tI) = \det \begin{bmatrix} 0 & t^2 & 0 & 0 & a_1t+a_0 \\ -1 & t & 0 & 0 & a_1 \\ 0 & -1 & t & 0 & a_2 \\ 0 & 0 & -1 & t & a_3 \\ 0 & 0 & 0 & -1 & t+a_4 \end{bmatrix}.$$

Similarly, if we now add t^2 times the third row to the first row, we get

$$\det(A+tI) = \det \begin{bmatrix} 0 & 0 & t^3 & 0 & a_2t^2+a_1t+a_0 \\ -1 & t & 0 & 0 & a_1 \\ 0 & -1 & t & 0 & a_2 \\ 0 & 0 & -1 & t & a_3 \\ 0 & 0 & 0 & -1 & t+a_4 \end{bmatrix}.$$

Now adding t^3 times the fourth row to the first row, then adding t^4 times the fifth row to the first row, gives

$$\det(A+tI) = \det \begin{bmatrix} 0 & 0 & 0 & 0 & t^5+a_4t^4+a_3t^3+a_2t^2+a_1t+a_0 \\ -1 & t & 0 & 0 & a_1 \\ 0 & -1 & t & 0 & a_2 \\ 0 & 0 & -1 & t & a_3 \\ 0 & 0 & 0 & -1 & t+a_4 \end{bmatrix}.$$

Expanding by minors along the first row, this is equal to

$$\begin{aligned} &(-1)^{1+5}(t^5+a_4t^4+\cdots+a_1t+a_0)\det \begin{bmatrix} -1 & t & 0 & 0 \\ 0 & -1 & t & 0 \\ 0 & 0 & -1 & t \\ 0 & 0 & 0 & -1 \end{bmatrix} \\ &= (-1)^6 \cdot (-1)^4(t^5+a_4t^4+\cdots+a_0) = t^5+a_4t^4+\cdots+a_0. \end{aligned}$$

For general n, we get $(-1)^{n+1}(-1)^{n-1}(t^n+a_{n-1}t^{n-1}+\cdots+a_1t+a_0) = t^n+a_{n-1}t^{n-1}+\cdots+a_1t+a_0$.

X1. ($\Rightarrow$) If A and A^{-1} both have integer entries, then $\det A$ and $\det(A^{-1})$ are both integers. However, we also have $(\det A)(\det(A^{-1})) = \det I = 1$; since the only way a product of integers can be 1 is if both are ± 1, this implies $\det A = \pm 1$.

($\Leftarrow$) Since A has integer entries, each determinant of an $(n-1)\times(n-1)$ minor is an integer, so the cofactor matrix C also has integer entries. Thus, if $\det A = \pm 1$ also, then $A^{-1} = \frac{1}{\det A}C^t$ also has integer entries.