Homework 5 Solutions

Math 110

Section 2.6

3. (a) Suppose $\beta^* = \{f_1, f_2, f_3\}$. Then if $f_1(x, y, z) = a_1x + b_1y + c_1z$, we must have

$$\begin{aligned} f_1(1, 0, 1) &= a_1 + c_1 = 1, \\ f_1(1, 2, 1) &= a_1 + 2b_1 + c_1 = 0, \\ f_1(0, 0, 1) &= c_1 = 0. \end{aligned}$$

Solving, we get $a_1 = 1$, $b_1 = -\frac{1}{2}$, and $c_1 = 0$, so $f_1(x, y, z) = x - \frac{1}{2}y$. Similarly, we get $f_2(x, y, z) = \frac{1}{2}y$, and $f_3(x, y, z) = -x + z$.

(b) Here, if $\beta^* = \{f_0, f_1, f_2\}$, we easily see that

$$\begin{aligned} f_0(a_0 + a_1x + a_2x^2) &= a_0, \\ f_1(a_0 + a_1x + a_2x^2) &= a_1, \\ f_2(a_0 + a_1x + a_2x^2) &= a_2. \end{aligned}$$

4. It is easy to see that if $x - 2y = x + y + z = y - 3z = 0$, then $x = y = z = 0$; by theorem (*) from lecture, this implies that $\{f_1, f_2, f_3\}$ spans $(\mathbb{R}^3)^*$ and is thus a basis since $\dim(\mathbb{R}^3)^* = 3$.

Now if $\{f_1, f_2, f_3\} = \{v_1, v_2, v_3\}^*$, suppose $v_1 = (x_1, y_1, z_1)$; then we must have

$$\begin{aligned} f_1(x_1, y_1, z_1) &= x_1 - 2y_1 = 1 \\ f_2(x_1, y_1, z_1) &= x_1 + y_1 + z_1 = 0 \\ f_3(x_1, y_1, z_1) &= y_1 - 3z_1 = 0. \end{aligned}$$

Solving, we get $x_1 = \frac{2}{5}$, $y_1 = -\frac{3}{10}$, $z_1 = -\frac{1}{10}$; thus $v_1 = (0.4, -0.3, -0.1)$. Similarly, $v_2 = (0.6, 0.3, 0.1)$ and $v_3 = (0.2, 0.1, -0.3)$.

7. (a) We have $(T^t f)(p) = f(Tp) = f(p(0) - 2p(1), p(0) + p'(0)) = -p(0) - 2p(1) - 2p'(0)$.

(b) Suppose $\gamma^* = \{p_1, p_2\}$ and $\beta^* = \{c_0, c_1\}$, where p_1, p_2 are the projection functions $\mathbb{R}^2 \to \mathbb{R}$, and $c_0, c_1 : P_1(\mathbb{R}) \to \mathbb{R}$ are the functions which take the coefficients of a polynomial. Then we have $(T^t p_1)(p) = p_1(p(0)-2p(1), p(0)+p'(0)) = p(0)-2p(1)$. Since this functional equals -1 when applied to $p(x) = 1$ and -2 when applied to $p(x) = x$, we get $T^t p_1 = -c_0 - 2c_1$ Similarly, $(T^t p_2)(p) = p_2(p(0) - 2p(1), p(0) + p'(0)) = p(0) + p'(0)$. This functional equals 1 when applied to $p(x) = 1$ and 1 when applied to $p(x) = x$, so $T^t p_2 = c_0 + c_1$. Therefore, we get

$$[T^t]_{\gamma^*}^{\beta^*} = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}.$$

(c) Since $T(1) = (-1, 1)$ and $T(x) = (-2, 1)$, we have

$$[T]_\beta^\gamma = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}.$$

The transpose of this matrix is exactly the matrix we calculated in the previous part.

11. Let $x \in V$; then since $\psi(x) = \hat{x}$, we need to show that $(Tx)\hat{} = T^{tt}(\hat{x})$. Since both sides are in W^{**}, suppose we have $g \in W^*$. Then

$$(Tx)\hat{}(g) = g(Tx).$$

On the other hand,

$$(T^{tt}(\hat{x}))(g) = \hat{x}(T^t g) = (T^t g)(x) = g(Tx).$$

Since this is true for any $g \in W^*$, this shows that $T^{tt}(\hat{x}) = (Tx)\hat{}$.

12. Suppose $\beta = \{x_1, x_2, \ldots, x_n\}$, and $\beta^* = \{f_1, f_2, \ldots, f_n\}$ is the dual basis. We need to show that $\psi(\beta) = \{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n\}$ is the dual basis to β^*. However, by definition, $f_i(x_j) = \delta_{ij}$ for each i, j. Therefore,

$$\hat{x}_i(f_j) = f_j(x_i) = \delta_{ji} = \delta_{ij}$$

for each i, j.

15. Let $g \in W^*$; then $g \in N(T^t)$ if and only if $T^t g = 0$, which is equivalent to the condition that $(T^t g)(x) = 0$ for each $x \in V$. However, since $(T^t g)(x) = g(Tx)$, this is equivalent to saying that $g(Tx) = 0$ for each $x \in V$, which is equivalent to the condition that $g(y) = 0$ for each $y \in R(T)$. However, this condition is exactly the definition of $g \in R(T)^0$.

16. First, let ε_n, ε_m be the standard bases of F^n, F^m, with dual bases ε_n^*, ε_m^*. Then we have

$$[L_A^t]_{\varepsilon_m^*}^{\varepsilon_n^*} = \left([L_A]_{\varepsilon_n}^{\varepsilon_m}\right)^t = A^t.$$

Therefore, $\operatorname{rank}(L_A^t) = \operatorname{rank}(L_{A^t})$, so we reduce to showing that for $T : V \to W$, with V, W finite-dimensional, we have $\operatorname{rank}(T^t) = \operatorname{rank}(T)$.

To see this, since $T^t \in L(W^*, V^*)$, by the dimension theorem, we have $\dim R(T^t) + \dim N(T^t) = \dim(W^*) = \dim W$. On the other hand, since $R(T)$ is a subspace of W, we also have $\dim R(T) + \dim R(T)^0 = \dim W$. Finally, from the previous problem, we have $\dim N(T^t) = \dim R(T)^0$. Putting these facts together, we conclude $\dim R(T^t) = \dim R(T)$.

X1. Since f is nonzero, $R(f) \neq \{0\}$; therefore, $R(f) = F$. Thus, $\dim N(f) = \dim V - \dim R(f) = \dim V - 1$.

X2. Since $TU \in L(V, X)$, $(TU)^t \in L(X^*, V^*)$; similarly, since $U^t \in L(W^*, V^*)$ and $T^t \in L(X^*, W^*)$, $U^t T^t \in L(X^*, V^*)$ also.

Now for $h \in X^*$, we need to show $(TU)^t h = U^t(T^t h)$; since both sides are in V^*, suppose $x \in V$. Then

$$[(TU)^t h](x) = h(TU(x)).$$

On the other hand,

$$[U^t(T^t h)](x) = (T^t h)(Ux) = h(T(Ux)) = h(TU(x)).$$

Therefore, $(TU)^t h = U^t(T^t h)$ for any $h \in X^*$, showing that $(TU)^t = U^t T^t$.

X3. Assuming that T is an isomorphism, we show that $(T^{-1})^t : V^* \to W^*$ is an inverse function to T^t, which will imply the desired result. However, by the previous problem, we see that

$$(T^{-1})^t \circ T^t = (T \circ T^{-1})^t = (\mathrm{id}_W)^t = \mathrm{id}_{W^*}\,.$$

Similarly,

$$T^t \circ (T^{-1})^t = (T^{-1} \circ T)^t = (\mathrm{id}_V)^t = \mathrm{id}_{V^*}\,.$$

X4. (a) Since $\dim V^* = 4$, it suffices to show that $\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3\}$ spans V^*. To do this we use theorem (*) from lecture; thus, suppose that $\varepsilon_0(p) = \varepsilon_1(p) = \varepsilon_2(p) = \varepsilon_3(p) = 0$ for some $p \in P_3(\mathbb{R})$. Then as in an example from lecture, since $p(0) = p(1) = p(2) = p(3) = 0$, $x(x-1)(x-2)(x-3)$ must divide $p(x)$, and since $\deg p \leq 3$, this implies $p = 0$.

(b) Suppose that $\varepsilon_4 = c_0\varepsilon_0 + c_1\varepsilon_1 + c_2\varepsilon_2 + c_3\varepsilon_3$. Then plugging in $p(x) = 1$, we get $1 = c_0 + c_1 + c_2 + c_3$. Similarly, plugging in $p(x) = x$, $p(x) = x^2$, and $p(x) = x^3$, we get the system of equations

$$\begin{aligned} c_0 + c_1 + c_2 + c_3 &= 1 \\ c_1 + 2c_2 + 3c_3 &= 4 \\ c_1 + 4c_2 + 9c_3 &= 16 \\ c_1 + 8c_2 + 27c_3 &= 27. \end{aligned}$$

Solving, we get $c_0 = -1$, $c_1 = 4$, $c_2 = -6$, and $c_3 = 4$. Therefore, $\varepsilon_4 = -\varepsilon_0 + 4\varepsilon_1 - 6\varepsilon_2 + 4\varepsilon_3$. Alternately, if we plug in $x(x-1)(x-2)$ in both sides, we get $24 = 6c_3$, so $c_3 = 4$; similarly, plugging in $x(x-1)(x-3)$, $x(x-2)(x-3)$, and $(x-1)(x-2)(x-3)$ gives c_2, c_1, c_0 directly.